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MATHEMATICAL EXPRESSION FOR THE NEWSVENDOR PROFITS

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ABSTRACT

The Newsboy Problem, or the more politically correct The Newsvendor Problem, is a classical OR problem pertaining to Inventory Theory. It arises in situations involving seasonal or perishable products that cannot be carried in inventory and sold in future periods at a profit. In this brief work, a closed expression for the profits expected by the newsvendor is obtained. Using partial expectations, an interesting result is obtained for the Poisson distribution. Some additional results for the Uniform, Normal and Exponential probability distributions are also obtained.

1) Problem Definition

The Newsboy Problem, or the more politically correct The Newsvendor Problem, is a classical OR problem pertaining to Inventory Theory. It arises in situations involving seasonal or perishable products that cannot be carried in inventory and sold profitably in future periods. A description of the simplest case may be found in [1] or in [2]. An updated description of the more general case may be found in [3]. The Newsvendor Problem is mainly applied in buying perishable products under a demand that has a random behavior.. Its goal is to obtain the quantity of product to be bought knowing that, if one under-buys, one would incur in lost sales. On the contrary, if one over-buys, one would have to sell the remainder product later-on at a discounted price. A newsvendor illustrates clearly this situation: It is important to be well-stocked to satisfy the customers but a newspaper not sold today has a negligible value tomorrow.

More formally, let:

x: Random variable that represents the demand of a product.. This variable is defined between an inferior value I and a superior one S.

f(x): Probability density function of the demand.[1]

F(x): Cumulative distribution function of the demand.

[1] In this work a price-dependent demand function is not treated. See the comment at the end. Also, only the continuous case is treated in the main text. The discrete case is presented in Appendix 1.

Q: Quantity of products to be acquired.

P: Profitable unit selling price of the product.

H: Discounted unit selling price of the product.

C: Unit cost of the product.

V: Lost-sales unit cost.

Of course, one assumes that:

$$f(x) \geq 0, \text{ for } I \leq x \leq S$$

$$\int_I^S f(x)\,dx = 1$$

$$f(x) = dF(x) / dx$$

and, $P > C > H$

2) Development

The objective of the Newsvendor Problem is to find the value of Q that maximizes the expected value of the profits.

Thus, let Q be the quantity of products to be acquired for later sale. The cost incurred would be:

Total cost: $C\,Q$

For a demand less or equal than Q, one would observe earnings of:

Profitable earnings: $P \int_I^Q x\, f(x)\,dx$

Non-profitable earnings: $H \int_I^Q (Q - x)\, f(x)\,dx$

Where $(Q - x)$ denotes the remainder product which could not be sold at a price P and has to be disposed at a discounted price H.

For a demand x greater than Q, one has assured earnings of:

Assured earnings: $P\,Q \int_{Q}^{S} f(x)\,dx$

Note that this term is equal to P Q (1 – F(Q)).

And a cost of lost-sales of:

Lost-sales loss: $V \int_{Q}^{S} (x - Q)\, f(x)\,dx$

Then, the expected profit U(Q), would be:

$$U(Q) = P \int_{I}^{Q} x\, f(x)\,dx + H \int_{I}^{Q} (Q - x)\, f(x)\,dx + P\,Q\,(1 - F(Q)) - V \int_{Q}^{S} (x - Q)\, f(x)\,dx - C\,Q$$

Which may be further simplified as:

$$U(Q) = (P - H) \int_{I}^{Q} x\, f(x)\,dx - V \int_{Q}^{S} x\, f(x)\,dx + H\,Q\,F(Q) + (P + V)\,Q\,(1 - F(Q)) - C\,Q$$

Finally, ordering terms and factorizing, one obtains:

$$U(Q) = (P - H) \int_{I}^{Q} x\, f(x)\,dx - V \int_{Q}^{S} x\, f(x)\,dx - (P + V - H)\,Q\,F(Q) + (P + V - C)\,Q \qquad (1)$$

To obtain the value of Q which optimizes (1) one follows the usual method:

$$dU(Q) / dQ = (P - H)\,Q\,f(Q) + V\,Q\,f(Q) - (P + V - H)\,[F(Q) + Q\,f(Q)] + (P + V - C)$$

Which equals:

$$dU(Q) / dQ = (P + V - H)\,[- F(Q)] + (P + V - C)$$

Making it equal to zero one obtains the well-known result:

$$F(\boldsymbol{Q}) = (P + V - C) / (P + V - H) = \boldsymbol{\alpha} \qquad (2)$$

Where $\boldsymbol{Q}$ denotes the optimal value of Q.

Up-to-now there is nothing new. However, if one replaces the value of F($\boldsymbol{Q}$) in (1), the expected profit just found, one obtains almost directly the following expression for the expected profit at the optimum:

$$U(\boldsymbol{Q}) = (P - H) \int_{I}^{Q} x\, f(x)\, dx \; - \; V \int_{Q}^{S} x\, f(x)\, dx \qquad (3)$$

In words, the optimal expected profit is the product of the difference of the profitable and the non-profitable prices multiplied by a partial expected value of the demand evaluated up-to $\boldsymbol{Q}$, less the lost-sales unit cost multiplied by the remainder of the expected value of the demand.

Note that if V = 0 and the newsvendor selects the optimal value $\boldsymbol{Q}$, the newsvendor would have a positive value of its expected profits.

3) <u>Optimal Density Function</u>

In Equation (3) one might ask which of the possible density functions f(x) maximizes the expected profit of the newsvendor. This function may be easily derived from the discrete treatment of this problem, treated in Appendix 1.

The solution is:

$$\boldsymbol{f}(\boldsymbol{Q}) = \boldsymbol{\alpha}$$

$$\boldsymbol{f}(\boldsymbol{Q} + \varepsilon) = \mathbf{1} - \boldsymbol{\alpha}$$

$$\boldsymbol{f}(x) = 0, \text{ elsewhere in } I <= x <= S.$$

Where ε is an arbitrary positive number.

In words, the optimal density function, denoted by $\boldsymbol{f}$ is that one that concentrates the density mass denoted by $\boldsymbol{\alpha}$ in Equation (2) in the optimal point $\boldsymbol{Q}$, and the remainder density $(\mathbf{1} - \boldsymbol{\alpha})$ in an arbitrarily close point denoted by $(\boldsymbol{Q} + \varepsilon)$.

Note that in this case Equation (3) is reduced to:

$$U(\boldsymbol{Q}) = (P - C)\, \boldsymbol{Q} \; - \; V\,(1 - \boldsymbol{\alpha})\, \varepsilon \qquad (4)$$

Therefore, if ε equals zero, the newsvendor profit would be equal to the profit that could be attained without randomness in the demand. To put it in another way, the newsvendor expected profits will be larger the more concentrated the probability density function of the demand is around the optimal point $\boldsymbol{Q}$.

4) Some Results

With some mathematical gymnastics (See Appendix 2), it may be shown that the optimal for the expected value of the newsvendor profits for demands with Poisson, Uniform, Exponential and Normal distributions have the following expressions:

For the Poisson distribution, with expected value $\boldsymbol{\lambda}$, and defined for integer positive values of the random variable x, zero included:

$$U(\boldsymbol{Q}) = \boldsymbol{\lambda}\,(P - C) \qquad (5)$$

From the later, note that the newsvendor may obtain a positive expected profit if it faces a demand with a Poisson distribution and selects the optimal quantity Q.

For the Uniform distribution, defined for values of the demand between $\mathbf{a} \le x \le \mathbf{b}$:

$$U(\boldsymbol{Q}) = [(P - C)\,(\mathbf{a} + \boldsymbol{Q}) - V\,(\mathbf{b} - \boldsymbol{Q})] / 2 \qquad (6)$$

For the Exponential distribution with parameter $\boldsymbol{\alpha}$ (mean value $\mathbf{1/\,\boldsymbol{\alpha}}$):

$$U(\boldsymbol{Q}) = (P - C)\,(\mathbf{1/\,\boldsymbol{\alpha}}) - (C - H)\,\boldsymbol{Q} \qquad (7)$$

And for the Normal Distribution with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma^2}$:

$$U(\boldsymbol{Q}) = (P - C)\,\boldsymbol{\mu} - (P - H + V)\,\boldsymbol{\sigma^2}\,f(\boldsymbol{Q}) \qquad (8)$$

For the case where V = P – C, that is, when the lost-sales unit cost is just the difference between the profitable price of the product and its cost, it may be shown that the newsvendor would have a positive expected value for its profits if the following conditions are met::

For a demand with a Uniform Distribution defined between $\mathbf{a} \le x \le \mathbf{b}$:

$$2\,\boldsymbol{Q} > (\mathbf{b} - \mathbf{a}) \qquad (9)$$

For a demand that follows an Exponential Distribution with parameter $\boldsymbol{\alpha}$ (mean value $\mathbf{1/\,\boldsymbol{\alpha}}$):

$$2\,\boldsymbol{Q} < F(\boldsymbol{Q}) / f(\boldsymbol{Q}) \qquad (10)$$

Finally, for a demand with a Normal Distribution with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma^2}$:

$$\boldsymbol{\mu} > 2\,\boldsymbol{\sigma^2}\,f(\boldsymbol{Q}) / F(\boldsymbol{Q}) \qquad (11)$$

5) Final Words

The case where there is a price-dependent demand function was treated exhaustively using the same methodology followed in this paper without obtaining any closed results. This topic is left open for future research.

REFERENCES

[1] Principles of Operations Research. Harvey M. Wagner. Prentice-Hall.

[2] Introduction to Operations Research. Frederick S. Hillier and Gerald J. Lieberman. Holden-Day Inc.

[3] “Pricing and the Newsvendor Problem: A Review with Extensions”. Nicholas C. Petruzzi and Maqbool Dada. (January 1998).

APPENDIX 1

DISCRETE CASE

In the discrete case the demand of the product only takes discrete positive values. These values have associated a discrete probability measure. A bit more formally:

D_i: i-th value of the demand. The index i may take values equal to 1, 2, ..., n. It is required that $D_{i+1} > D_i$.

q_i: Probability of observing the demand D_i for the product.

F: Cumulative distribution function of the demand.

Q: Quantity of products to be acquired.

P: Profitable unit selling price of the product.

H: Discounted unit selling price of the product.

C: Unit cost of the product.

V: Lost-sales unit cost.

It is required that,

$q_i \geq 0$, for all i.

$$\sum_{i=1}^{n} q_i = 1$$

$$F_k = \sum_{i=1}^{k} q_i$$

P > C > H.

Then, let Q be the quantity of products to be acquired for selling to the customers. Total acquisition costs would be:

Total acquisition cost: $C\,Q$

For a demand D_i, less than or equal to Q one would expect earnings of::

Profitable earnings: $P \sum_{i=1}^{j} D_i q_i$

Non-profitable earnings: $H \sum_{i=1}^{j} (Q - D_i)\, q_i$

For a demand D_i greater than Q one has assured expected earnings of:

Assured earnings: $P\,Q \sum_{i=j+1}^{n} q_i$

This term is equal to $P\,Q\,(1 - F_j)$.

And a lost-sales expected cost of::

Lost-sales expected cost: $V \sum_{i=j+1}^{n} (D_i - Q)\, q_i$

Thus, expected profits, denoted by U(Q), would be:

$$U(Q) = (P - H) \sum_{i=1}^{j} D_i\, q_i \; - \; V \sum_{i=j+1}^{n} D_i\, q_i - (P + V - H)\, Q\, F_j + (P + V - C)\, Q \qquad (1)$$

Which may be differentiated with respect to Q to obtain by the customary optimization procedure:

$$F_j = (P + V - C) / (P + V - H) = \boldsymbol{\alpha} \qquad (2)$$

Finally, plugging this value into Equation (1) one obtains:

$$U(\boldsymbol{Q}) = (P - H) \sum_{i=1}^{j} D_i\, q_i \; - \; V \sum_{i=j+1}^{n} D_i\, q_i \qquad (3)$$

Now, the left-hand term of this equation is positive, because by definition $P > H$ and both D_i and q_i are positive. Focusing on q_i, by linearity this term attains a maximum for $q_1 = 0$, $q_2 = 0$, ..., $q_{j-1} = 0$ and $\mathbf{q_j = \boldsymbol{\alpha}}$, given that $D_j > D_{j-1} > D_{j-2} > \ldots > D_1$.

In the same fashion, the right-hand term of the equation (including the minus sign) is negative by definition. This term attains a minimal value if $\mathbf{q_{j+1} = (1 - \boldsymbol{\alpha})}$, given that $D_{j+1} < D_{j+2} < D_{j+3} < \ldots < D_n$.

Then, for given values of P, C, H y V, the probability set that maximizes the newsvendor expected profit is that one which concentrates **α** in the **j-th** value of the demand and **(1 – α)** in the next value, where **j** may be obtained form Equation (2).

APPENDIX 2

PARTIAL EXPECTED VALUES

In the main text the following equation appears:

$$U(Q) = (P - H)\int_{I}^{Q} x\, f(x)\, dx \; - \; V\int_{Q}^{S} x\, f(x)\, dx \qquad (1)$$

Where $I \leq Q \leq S$.

The integrals may be interpreted as partial expected values given that, E(x), the expected value of a probability distribution, may be split into:

$$E(x) = \int_{I}^{S} x\, f(x)\, dx = \int_{I}^{Q} x\, f(x)\, dx + \int_{Q}^{S} x\, f(x)\, dx$$

If the first term of the sum above is denoted by $E_Q(x)$, then:

$$E(x) = E_Q(x) + (E(x) - E_Q(x))$$

Therefore, Equation (1) may be expressed as:

$$U(Q) = (P - H)\, E_Q(x) - V\,(E(x) - E_Q(x)) = (P + V - H)\, E_Q(x) - V\, E(x) \qquad (2)$$

Or, alternatively, taking into account that:

$$F(Q) = (P + V - C) / (P + V - H)$$

Thus,

$$U(Q) = (P + V - C)\, E_Q(x) / F(Q) - V\, E(x) \qquad (3)$$

In the following, partial expected values are derived for the Poisson, Uniform, Exponential and Normal distributions.

<u>Poisson Distribution.</u>

This is a discrete distribution. Its expected value is:

$$E(x) = \sum_{i=0}^{\infty} \frac{i \quad e^{-\lambda} \quad \lambda^{i}}{i!} = \lambda$$

The partial expected value of this distribution may be calculated as follows:

$$E_j(x) = \sum_{i=0}^{j} \frac{i \quad e^{-\lambda} \quad \lambda^i}{i!} = \sum_{i=1}^{j} \frac{e^{-\lambda} \quad \lambda^i}{(i-1)!} = \sum_{s=0}^{j} \frac{e^{-\lambda} \quad \lambda^{s+1}}{s!}$$

$$E_j(x) = \lambda \sum_{s=0}^{j} \frac{e^{-\lambda} \quad \lambda^s}{s!} = \lambda F_j$$

Where F_j is the cumulative distribution evaluated from x = 0 up to x = j.

The complement of this partial expectation is just $\lambda (1 - F_j)$.

Uniform Distribution.

The expected value for the Uniform Distribution, defined for $a \leq x \leq b$ is:

$$E(x) = \frac{a+b}{2}$$

The expected value evaluated up to a value Q is:

$$E_Q(x) = \frac{1}{(b-a)} \int_a^Q x dx = \frac{Q^2 - a^2}{2(b-a)} = \frac{(Q+a)F(Q)}{2}$$

Since, for the Uniform Distribution:

$$F(Q) = \frac{Q-a}{b-a}$$

The complement of this partial expected value is:

$$E(x) - E_Q(x) = \frac{b^2 - Q^2}{2(b-a)}$$

Exponential Distribution.

This distribution is defined for positive values of the random variable (zero included). It has a probability density function and a cumulative distribution function defined respectively as:

$$f(x) = \alpha e^{-\alpha x}$$

$$F(x) = 1 - e^{-\alpha x}$$

Its expected value is $E(x) = 1 / \alpha$.

The partial expected value for this distribution is:

$$E_Q(x) = \int_0^Q x\, f(x)\, dx = \frac{1 - e^{-\alpha Q} - \alpha Q e^{-\alpha Q}}{\alpha} = \frac{F(Q) \quad - Q f(Q)}{\alpha}$$

Normal Distribution.

The well-known Normal Distribution, with parameters μ y σ^2, has a partial expected value of:

$$E_Q(x) = \int_{-\infty}^{Q} x\, f(x)\, dx = \mu\, F(Q) - \sigma^2\, f(Q)$$

Where f(x) and F(x) are the probability density function and the cumulative distribution function, respectively.

The partial expected value may be found quite easily if the following transformation is made:

$$E_Q(x) = \int_{-\infty}^{Q} x\, f(x)\, dx = \sigma \int_{-\infty}^{Q} \frac{\mu}{\sigma}\, f(x)\, dx + \sigma \int_{-\infty}^{Q} \frac{(x-\mu)}{\sigma}\, f(x)\, dx$$