A Multistage Stochastic Inventory Model in the Presence of Commodity Futures Market

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ABSTRACT

The proliferation of commodity markets in recent years has provided manufacturers an extra flexibility to deal with price uncertainty in raw material procurement. Although manufacturers have been using commodity derivative instruments for risk hedging for decades, such hedging programs are conducted purely as a financial exercise and are usually separated from manufacturers’ inventory planning and production activities. This paper describes the optimal inventory planning problem of make-to-order manufacturers in the presence of commodity futures market. By purchasing in the spot market and hedging in the commodity futures market, such a manufacturer will aim at minimizing the overall procurement cost while fulfilling multistage stochastic customer demand. Schwartz’s two-factor model is applied to explore the stochastic behaviour of commodity prices, and the inventory planning problem is formulated as a multistage stochastic program. Numerical simulations of the Schwartz’s two factor model and the proposed multistage stochastic inventory model are also presented in this paper.

Keywords: inventory management, stochastic programming, risk hedging, commodity futures

1. INTRODUCTION

Effective inventory management is crucial to the success of most manufacturing companies. An optimal inventory replenishment policy is desired by them to match closely the supply and demand at the lowest cost and risk. Unfortunately, by entering into long-term contracts with established suppliers to secure supply or to fix their procurement cost, such a manufacturer will be unable to control the level of input inventory. Due to the increasingly volatile market and ever decreasing profit margin, however, a manufacturer holding high level of inventory quite often suffers from unexpected price change, high inventory holding cost, risk of obsolescence, and inefficient use of working capital.

In recent years, the advancement of information technology and e-commerce has brought about major changes in the raw material procurement and sourcing environment manifested by a proliferation of online commodity markets. Today, commodity exchange does not only offer trading in traditional commodities (e.g. such as foodstuff, industrial metals and fuels), it also provides new derivative products for trading non-traditional commodities like paper, plastics, electricity and DRAM. The development of online commodity markets provides manufacturers an additional flexibility for hedging price fluctuations. Although manufacturers have been using commodity futures for price risk hedging for decades, in most cases these hedging programs are conducted purely as a financial exercise and are separated from the inventory replenishment planning activity. Also, most inventory models in the literature assume constant or known purchasing price and are not able to deal with commodity futures hedging. The development of optimal inventory replenishment plan in an environment with online commodity markets is challenging, especially for make-to-order manufacturers dealing with multi-period stochastic demands. Under such a new procurement environment these manufacturers now have to match closely two stochastic variables, the procurement price as well as the customer demand. Inventory replenishment planning model with the availability of online commodity market has been largely unexplored. Seifert and Hausman¹ seek to quantify the role of spot markets in the procurement process in which a risk-averse manufacturer determines the quantity allocation between a traditional manufacturer and the spot-market. They model price and demand as a bivariate normal distribution and illustrate that spot markets become increasingly relevant as the variance in demand increases. So far, the most thorough study of such problems is probably due to Goel and Gutierrez². By assuming access to futures market, commodity futures price is modelled using the two-factor model of Schwartz¹ and this model is solved as a stochastic dynamic
Another work can be found in Medova and Sembo[4], where they seek optimal price protection strategies for a consortium oil company facing uncertain demand and price by linking logistics planning with futures hedging. Their stochastic programming formulation of hedging provides a dynamic financial analysis technique for determining the optimal policy under uncertainty involved in future price movements.

The work described in this paper extends the traditional inventory management planning model by incorporating commodity futures contracts in the procurement of raw materials. Its approach bears some degree of resemblance to that of Goel and Gutierrez[2] and considers a make-to-order manufacturer with access to both spot and futures market of raw materials. This manufacturer is assumed to be a price taker, i.e. his procurement decision cannot affect market equilibrium. As a risk hedger but not a speculator, this manufacturer aims primarily at hedging the risk of high purchasing prices of future raw material and to minimize the procurement cost with the inventory holding cost and backorder penalty taken into consideration. Sometimes, however, a hedger might turn into a speculator when opportunity arises due to excessive speculation is present in the market. Therefore, futures contracts will only be used for the purpose of hedging but actual procurement takes place in the spot market. However, as different from Goel and Gutierrez[2], this manufacturer is assumed to be able to buy only in the spot market. Selling is only allowed at the end of the planning horizon at the spot price. This manufacturer experiences a multi-period stochastic demand for raw materials (e.g. copper or other metals that can be purchased from the London Metal Exchange, or LME). The objective of this paper is thus to determine the optimal inventory replenishment plan with uncertainty in both purchasing price and quantity. Instead of using stochastic dynamic programming technique, we differentiate our research from Goel and Gutierrez[2] in that we formulate a stochastic programming model to solve this problem.

The remainder of this paper is organised as follows. In section 2 a two-factor model of stochastic behaviour of commodity prices proposed by Schwartz[4] is studied. In section 3 a mean reverting stochastic process of demand is discussed. The decision making problem under uncertainties is formulated as a multistage stochastic program in section 4. Section 5 describes numerical simulations of the Schwartz’s two factor model and the multistage stochastic inventory model. The conclusion is presented in section 6.

2. SCHWARTZ’S TWO-FACTOR MODEL

The stochastic behaviour of commodity prices plays a central role in the stochastic programming formulation of the inventory replenishment planning problem considered in this paper. The approach described will follow the two-factor model proposed in Schwartz[4], who uses a continuous time model to characterize the spot price and instantaneous convenience yield. According to this model, the logarithmic of the spot price of the commodity follow an Ornstein-Uhlenbeck stochastic process while convenience yield observes a mean reverting stochastic process. The assumption of logarithmic spot price is made to ensure non-negative prices. The stochastic process of spot price and convenience yield are stated as:

\[
\begin{align*}
    dS &= (\mu - \delta)Sdt + \sigma_1 Sdz_1 \\
    d\delta &= \kappa(\alpha - \delta)dt + \sigma_2 dz_2
\end{align*}
\]

(1)

(2)

Where the increments to standard Brownian motion are correlated with:

\[dz_1 dz_2 = \rho dt\]

(3)

Defining \(X = \ln S\) and applying Ito’s Lemma to equation (1), the process for the logarithmic spot price can be written as:

\[
    dX = \left(\mu - \delta - 0.5\sigma_1^2\right)Sdt + \sigma_1 dz_1
\]

(4)

In this model the commodity is treated as an asset that pays a stochastic dividend yield \(\delta\). Thus, the risk adjusted drift of the commodity spot price process will be \(r - \delta\). Since convenience yield risk cannot be hedged, the risk-adjusted convenience yield process will have a constant market price of risk \(\lambda\) associated with it. The stochastic process of the spot price and convenience yield under equivalent martingale measure can be expressed as:

\[
\begin{align*}
    dS &= (r - \delta)Sdt + \sigma_1 Sdz_1^* \\
    d\delta &= \kappa(\alpha - \delta) - \lambda dt + \sigma_2 dz_2^* \\
    dz_1^* dz_2^* &= \rho dt
\end{align*}
\]

(5)

(6)

(7)

For many commodities, the spot price and instantaneous convenience yield are unobservable. In such cases, the state-space form and the Kalman filter[5] can be applied to estimate the parameters of the model and the time
series of the unobservable state variables. For the two-factor model, the measurement equation can be written as:

\[
y_t = d_t + Z_t \left[ X_t, \delta_t \right]' + \varepsilon_t, \quad t \in \{1, \ldots, T \}
\]

where

\[
y_t = \ln F(t), \quad i = 1, \ldots, N, \quad N \times 1 \text{ vector of observables}
\]

\[
d_t = A(t), \quad i = 1, \ldots, N, \quad N \times 1 \text{ vector}
\]

\[
A(t) = \left( r - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\sigma_1 \sigma_2 \rho}{\kappa} \right) \tau_i + \frac{\sigma_2^2 (1 - e^{-2\kappa\tau})}{4\kappa^4}
\]

\[
+ \left( \hat{\alpha} \kappa + \sigma_1 \sigma_2 \rho - \frac{\sigma_2^2}{\kappa} \right) \frac{1 - e^{-\kappa\tau}}{\kappa^2}
\]

\[i = 1, \ldots, N, \quad N \times 1 \text{ vector, } \hat{\alpha} = \alpha - \lambda / \kappa
\]

\[
Z_t = \begin{bmatrix} 1, -\frac{1 - e^{-e^{\tau_i}}}{\kappa} \end{bmatrix}, \quad i = 1, \ldots, N, \quad N \times 2 \text{ vector}
\]

\[\tau_i, \quad \text{Time to maturity of the } i\text{th observation}
\]

\[\varepsilon_t, \quad N \times 1 \text{ vector of serially uncorrelated disturbances with}
\]

\[E(\varepsilon_t) = 0, \quad \text{and } Var(\varepsilon_t) = H
\]

The transition equation can be written as:

\[
\begin{bmatrix} X_t, \delta_t \end{bmatrix}' = c_t + Q_t \left[ X_{t-1}, \delta_{t-1} \right]' + \eta_t, \quad t \in \{1, \ldots, T \}
\]

where

\[
c_t = \left[ \mu - 0.5 \sigma_1^2 \Delta t, \kappa \alpha \Delta t \right], \quad 2 \times 1 \text{ vector}
\]

\[
Q_t = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 - \kappa \Delta t \end{bmatrix},
\]

\[\eta_t, \quad \text{serially uncorrelated disturbances with}
\]

\[E(\eta_t) = 0, \quad \text{and } Var(\eta_t) = \begin{bmatrix} \sigma_1^2 \Delta t & \rho \sigma_1 \sigma_2 \Delta t \\ \rho \sigma_1 \sigma_2 \Delta t & \sigma_2^2 \Delta t \end{bmatrix}
\]

3. DEMAND PROCESS

A popular approach to model demand as is found in the literature is to assume that the demand follows a mean reverting random walk.

\[
d_{t+\Delta t} = e^{-\alpha \Delta t} (d_t - \bar{d}) + \bar{d} + \sigma' \sqrt{\Delta t} \theta
\]

where

\[
d_{t+\Delta t}, \quad \text{Demand at time } t + \Delta t
\]

\[
d_t, \quad \text{Demand at time } t
\]

\[
\bar{d}, \quad \text{Equilibrium demand}
\]

\[
\sigma', \quad \text{Mean reversion rate}
\]

\[
\sigma', \quad \text{Volatility of demand}
\]

\[
\theta, \quad \text{Disturbance with standard normal distribution}
\]

4. MULTISTAGE STOCHASTIC INVENTORY MODL

This paper considers a make-to-order manufacturer facing uncertainties arising from both raw material quantity and purchasing price. Instead of relying on long-term supply contracts, this manufacturer buys in the spot
market while uses commodity futures contracts to hedge unfavourable purchasing price movement. At each
decision making stage, this manufacturer need to decide how many units to buy in the spot market and the
outstanding position in futures market. For simplicity, we assume that at any stage there is an available futures
contract that expires at the next stage. To hedge over a longer time period, it is possible for the manufacturer to
roll over short-term contracts stage by stage. It is assumed that the raw material purchased in the spot market
will be delivered immediately after order placement, but before the observation of the demand. Another
assumption is that this manufacturer cannot sell in the spot market, except that at the end of the planning horizon
ending inventory can be sold at a salvage value. There is a holding cost for excess inventory and a backorder
penalty for unfulfilled demand. At the last stage if there are excess futures contracts outstanding a penalty cost
will be incurred. The objective is to maximize the manufacturer’s total profit. The variables and parameters used
in the stochastic model are defined as follows.

**Decision variables:**

\[x_t: \text{Units of raw material bought in the spot market. } t \in \{1,\ldots, T-1\}\]

\[u_t: \text{Outstanding position in the futures market. } t \in \{1,\ldots, T-1\}\]

**State variables:**

\[I_t: \text{Raw material inventory. } t \in \{2,\ldots, T\}\]

\[I_t^+: \text{Physical raw material inventory. } t \in \{2,\ldots, T\}\]

\[I_t^-: \text{Backorder of unfulfilled demand. } t \in \{2,\ldots, T\}\]

\[O: \text{Outstanding position in the futures market at stage } T\]

**Parameters:**

\[s_t: \text{Raw material spot price. } t \in \{2,\ldots, T\}\]

\[f_t^+: \text{Future price at stage } t \text{-} t \text{ with delivery at stage } t. \ t \in \{2,\ldots, T\}\]

\[d_t: \text{Demand for raw material. } t \in \{2,\ldots, T\}\]

\[U_t: \text{Bound of outstanding position in the futures market. } t \in \{1,\ldots, T\}\]

\[h: \text{Inventory holding cost.}\]

\[b: \text{Backorder penalty cost.}\]

\[v: \text{Price for selling excess ending inventory}\]

\[T: \text{Number of decision making stages}\]

\[\Delta t: \text{Time interval of stages}\]

\[r: \text{Risk free interest rate}\]

\[B: \text{Bound on the number of units bought in the spot market at each stage}\]

\[p: \text{Price for selling finished product}\]

\[w: \text{Penalty cost of excess hedging in the futures market}\]

To formulate the objective function, we define the revenue as

\[R(I^+, I^-) = p(d_t - I_t^-)e^{-rt} + \sum_{t=2}^{T} p(d_t + I_{t-1}^- - I_t^-)e^{-r(t-1)\Delta t} + vI_t^+ e^{-r(T-1)\Delta t}\]  \hspace{1cm} (11)

and the expense as

\[E(I^+, I^-, x, u) = \sum_{t=2}^{T} (hI_t^+ + bI_t^- + s_t x_{t-1} - (s_t - f_t^+) u_{t-1})e^{-r(t-1)\Delta t} + wO e^{-r(T-1)\Delta t}\] \hspace{1cm} (12)

The multistage stochastic inventory management problem can be modelled as:

\[\text{maximize } R(I^+, I^-) - E(I^+, I^-, x, u)\]  \hspace{1cm} (13a)

s.t.

\[I_t = I_t^+ - I_t^- , \quad \forall t \in \{2,\ldots, T\}\] \hspace{1cm} (13b)

\[I_t = \begin{cases} x_{t-1} - d_t, & \text{if } t = 2 \\ I_{t-1} + x_{t-1} - d_t, & \forall t \in \{3,\ldots, T\}\end{cases}\] \hspace{1cm} (13c)

\[O = \max\{u_{T-1} - x_{T-1}, 0\}, \quad \forall t \in \{2,\ldots, T\}\] \hspace{1cm} (13d)
5. NUMERICAL SIMULATION

This section illustrates the proposed approach with an example of a make-to-order manufacturer who uses copper as raw material in production. Instead of entering into traditional long-term supply contracts with established copper suppliers, the manufacturer buys copper in spot markets, and at the same time takes a position in London Metal Exchange (LME) copper futures to hedge unfavourable copper price change. For simplicity, it is assumed that the manufacturer trades only the 3 month copper futures contracts, although there are other contracts with longer maturity. It is also assumed the time interval between two stages is 3 month, and so futures contracts bought at a stage will be expired at the next immediate stage.

5.1 SIMULATION OF SCHWARTZ’S TWO FACTOR MODEL

The sample data used for simulating the stochastic behaviour of copper prices is the Friday price of LME 3 month copper futures contracts from 14/06/1991 to 06/12/2002. There are totally 600 observations as shown in the Figure 1. Since new 3 month copper futures contracts are issued daily, the time to maturity in this simulation remains unchanged, i.e. 3 months, which greatly facilitates the calculation. This price data can be obtained from http://www.econstats.com/.

Using Kalman Filtering the parameters of the Schwartz’s two factor model can be estimated. The results are listed in Table 1. Figure 2 displays the estimated state variables (the logarithm of the spot price and the instantaneous convenience yield) and the logarithm of the 3 month futures price over the sampling period. Figure 3 shows the prediction error of the futures price. As can be seen, the prediction error is low.
Table 1: Result of Parameter Estimation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Appr.Std.Dev.</th>
<th>t-test</th>
<th>Gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.3258</td>
<td>0.0017</td>
<td>191.3464</td>
<td>181.3736</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>1.1559</td>
<td>0.0018</td>
<td>648.6858</td>
<td>41.2489</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.2489</td>
<td>0.0017</td>
<td>149.1387</td>
<td>160.9556</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.2663</td>
<td>0.0013</td>
<td>200.5735</td>
<td>-187.9504</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0.2867</td>
<td>0.0012</td>
<td>241.4099</td>
<td>159.6905</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>120.8204</td>
<td>183373.3124</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.8208</td>
<td>0.0017</td>
<td>474.9330</td>
<td>20.9139</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.2564</td>
<td>0.0017</td>
<td>152.4243</td>
<td>14.8754</td>
</tr>
</tbody>
</table>

Figure 2: Estimated state variables (the logarithm of the spot price and the instantaneous convenience yield) and the logarithm of the 3 month futures price for each week from 14/06/1991 to 06/12/2002.
5.2 SIMULATION OF COPPER DEMAND
As discussed in section 3, the copper demand follows a mean reverting stochastic process defined in equation (10). In this example, we assume $\alpha' = 3$, $\bar{d} = 100$, $\sigma' = 20$, $d_1 = 100$, $r = 0.06$, $\Delta t = 1/4$. Figure 4 shows 50 sampling paths of the copper demand.

5.3 IMPLEMENTATION OF MULTISTAGE STOCHSTIC INVENTORY MODEL
In this section, a multistage stochastic inventory model is implemented in Xpress-SP. The parameters and random values of spot price, futures price and copper demand are stored in data files to separate the model and
the data. Here we consider $T = 5$, $h = 100$, $b = 300$, $v = 2000$, $p = 5000$, $r = 0.06$, $w = 50$. For simplicity, it is assumed that from each node there are 10 branches, which generates a scenario tree with 1, 10, 100, 1000, 10000 nodes in the first, second, third, fourth and the fifth stage of the scenario tree respectively.

The Xpress-SP implementation of the multistage stochastic inventory model is shown below.

```plaintext
! ------------------------------------------- lme.mos -------------------------------------------
model lme
! model name
uses 'mmsp', 'mmsystem'
! mosel library for stochastic programming

parameters
DATAFILE="lme_xpress.dat"
! parameters and scenarios
end-parameters

declarations
T:integer ! number of stages
nScen:integer ! number of scenarios
r:real ! risk free rate
DT:real ! time interval between two stages
end-declarations

initializations from DATAFILE
T
nScen
r
DT
end-initializations

declarations
Stages=1..T
N:array(1..T) of integer ! number of nodes at each stage
Mmax:array(1..T-1) of real ! upper bounds on u
NodeNum:array(1..T,1..T) of integer ! node number mat
D,F,S:array(2..T) of sprand ! D-demand; S-spot price; F-futures price
NodeD,NodeF,NodeS, NodeProb:array(2..T,1..nScen) of real
p:real ! selling price of finished goods
h:real ! inventory holding cost
b:real ! backlogging/shortage cost
v:real ! salvage value of finished goods
w:real ! penalty cost of outstanding futures at the last stage
Bmax:real ! upper bound on x
u:array(1..T-1) of spvar ! outstanding futures in stage t, to be delivered at next stage
x:array(1..T-1) of spvar ! spot market buying, immediate delivery, before demand realization
I:array(2..T) of spvar ! net inventory at stage t
I_p:array(2..T) of spvar ! physical inventory at stage t
I_m:array(2..T) of spvar ! backorder at stage t
R:splinctr ! total revenue
E:splinctr ! total expenses
P:splinctr ! total profit
NetInv:array(2..T) of splinctr ! net inventory balance
InvBal:array(2..T) of splinctr ! inventory balance across stages
OutstFt:splinctr ! outstanding futures constraint at stage T
MaxOutstBuy:array(1..T-1) of splinctr ! bound on outstanding futures
MaxSpotBuy:array(1..T-1) of splinctr ! bound on spot buying
C:splinctr ! penalty of outstanding futures at the last stage
O_p:spvar ! outstanding futures at the last stage
O_m:spvar ! units not hedged at the last stage
end-declarations

setparam('xsp_implicit_stage',true)
setparam('xsp_disp_warnings',false)
! don't display warning messages
spsetstages(Stages)
! explicitly set stage
```
!spsetstage(uT_m,T-1)   !explicitly set stage

initializations from DATAFILE
   Mmax  N  NodeNum  NodeD  NodeS
   NodeF  NodeProb  p  h  b  v  w  Bmax
end-initializations

spcreatetree(NodeNum)
forall(t in 2..T, n in 1..N(t)) do
   spsetrandatnode (F(t), n, NodeF(t,n))
   spsetrandatnode (S(t), n, NodeS(t,n))
   spsetrandatnode (D(t), n, NodeD(t,n))
   spsetprobcondatnode (t,n, NodeProb(t,n))
end-do

!----------------------------------------------Model formulation -------------------------------
forall(t in 2..T) I(t) is_free !set as free variable
R:= p*(D(2)-I_m(2))exp(-r*DT)+
   sum(t in 3..T) p*(D(t)+I_m(t-1)-I_m(t))*exp(-r*(t-1)*DT)+
   v*I_p(T)*exp(-r*(T-1)*DT)
C:= w*O_p*exp(-r*(T-1)*DT)
E:= sum(t in 2..T) (h*I_p(t)+b*I_m(t)+S(t)*x(t-1))*exp(-r*(t-1)*DT) +C
P:=R-E
forall(t in 2..T) NetInv(t):=I(t)-I_p(t)-I_m(t)
forall(t in 2..T)
   if (t>2) then InvBal(t):=I(t)=I(t-1)+x(t-1)-D(t);
   else InvBal(t):=I(t)=x(t-1)-D(t)
end-if
forall(t in 1..T-1) MaxOutstFt(t):=u(t)<=Mmax(t)
forall(t in 1..T-1) MaxSpotBuy(t):=x(t)<=Bmax
!OutstFt:=O_p=O_m+u(T-1)-x(T-1)
setparam('xsp_scen_based',false)
maximize(P) !maximize the expected profit
writeln("Optimal Objective Found: " + speval(P))
!writeln("Time Used: ",gettime-starttime,"s")
end-model

6. CONCLUSION

This paper describes a study of the optimal inventory replenishment planning problem with the presence of online commodity market. A multistage stochastic programming formulation has been developed to model the replenishment planning problem for a make-to-order manufacturer facing uncertainty in both raw material purchasing price and demand. The contribution of this paper is to incorporate futures hedging in the inventory control decision making process and to provide an opportunity for manufacturers to reduce procurement related cost.

REFERENCES