Over the past three years, increased rates of mortgage foreclosures in the U.S. have had devastating impacts on individuals, communities, organizations and government. In this paper we present dynamic and stochastic programming models to assist community-based organizations in the allocation of resources to foreclosed properties for acquisition and redevelopment, taking into account uncertain market conditions that change over time. Specifically, we extend previous results in the literature to include some special case models that do not contain the restrictive assumption of dominance relationships.

Keywords
foreclosure, portfolio management, dynamic programming, stochastic programming, resource allocation.

1 Introduction

A dramatic increase in mortgage foreclosures has had adverse effects in all sectors of the economy but especially in the housing sector since 2006. In response, non profit community development corporations (CDCs) provide a variety of services to mitigate these effects of foreclosures. These organizations which acquire foreclosed properties to support neighborhood stabilization and revitalization, take many actions on foreclosed properties to minimize blight, and provide affordable housing opportunities. However, the costs of all these actions exceed the limited resources available to typical community-based organizations. Thus, foreclosed housing property acquisition for neighborhood stabilization and revitalization becomes a significant dynamic decision problem for these organizations.

The foreclosed housing acquisition problem is a type of resource allocation problem where a limited budget is allocated dynamically to maintain an optimal portfolio of acquired properties. The problem involves stochasticity due
to the uncertainty in the returns of the properties and their availability based on the conditions of the housing market and foreclosures. There is substantial research literature on stochastic and dynamic models for resource allocation and portfolio management. A few examples that are somewhat related to our problem include [3], where the authors develop a decision model for the allocation of resources to road maintenance activities by combining several criteria in a single objective function. [2] develops a dynamic model for allocating resources to different new product development projects and identify analytical solutions by considering different types of return functions. In addition, [4] describes a stochastic programming approach to project portfolio management in the presence of endogenous uncertainty. However, no published work is known that specifically addresses decision modeling for foreclosed housing acquisition in a stochastic setting, except [1], where the authors provide a more detailed introduction to the problem.

In this paper, we first describe a dynamic programming formulation to model the foreclosed housing acquisition problem. This formulation and a special case which is referred to as stochastic dominancy relationship are also discussed in detail in [1]. We also present the stochastic programming model that can be used for the numerical analysis of the problem.

Main contributions of this research are as follows: (1) Our problem differs from classical resource allocation and portfolio management problems in that the set of properties that can be invested in is stochastic. (2) We extend previous results in [1] to include models without the restrictive assumption of dominance relationship. Moreover, our objective function involves several social criteria hence differ from classical cost based objectives used for resource allocation. We combine these multiple social criteria into a utility based single objective function.

The remainder of this paper is structured as follows: The stochastic optimization models are described in Section 2.1, while the structure of the optimal policy under some special cases is discussed in Section 3. In Section 2.2, we present the multistage stochastic programming model for the general housing acquisition problem. Finally our conclusions are outlined in Section 4.

2 The Stochastic Optimization Models

We present two stochastic optimization models for the foreclosed housing acquisition problem: a dynamic programming model and a multi-stage stochastic programming model.

2.1 The Dynamic Programming Model

We assume that a community based organization can acquire a set of properties from a set \( \mathcal{N} \) of available units, where \( \mathcal{N} \) is partitioned into a set of categories \( i = 1, \ldots, |I| \). We consider a finite planning horizon consisting of multiple decision periods \( t = \{1, 2, \ldots, T\} \) with \( B_t \) and \( c_t \) being the budget and the acquisition cost in period \( t \), respectively.

We assume that properties in each category have the same characteristics which are stochastic and dependant on
external market environments. To this end, we let the random vector \( \mu_t \) represent the estimates of the returns from the acquisitions. The return from an acquisition is assumed to be realized upon acquisition, and are accrued additively in each period. In our model, we assume that market conditions are stable with respect to the distribution of the return values, such that \( E[\mu_t - \mu_{t-1}] = 0 \).

We denote the number of units available in each category in period \( t \) by the vector \( Q_t \), and the total number of units acquired by period \( t \) by the vector \( \phi_t \). Hence \( \mu_t, Q_t \) and \( \phi_t \) represent the state vectors at period \( t \). Given these values, the organization decides on an acquisition vector \( x_t \), representing the number of units to acquire from each category.

The return value \( \mu_t \) transitions to \( \mu_{t+1} + \varepsilon_{\mu_t} \) where \( \varepsilon_{\mu_t} \) represents some random exogenous effect. Similarly, we model the transition to \( Q_{t+1} \) as \( Q_t - x_t + \varepsilon_{Q_t} \), where the last term represents the addition or removal of units from each category due to exogenous effects, i.e. purchases by other parties, new foreclosures, etc. Finally, \( \phi_{t+1} \) is determined by a deterministic mapping such that \( \phi_{t+1} = \phi_t + x_t \). The described decision process can be modeled as a dynamic programming problem that satisfies the following Bellman equations:

\[
V_t = \max_{x_t \in X_t} \{ \mu_t \phi_t + (\mu_t - c_t)x_t + E[V_{t+1}(\mu_{t+1}, \phi_{t+1}, Q_{t+1})]\} \quad t = 1, \ldots, T - 1
\]

\[
V_T = \mu_T \phi_T
\]

No discounting is modeled in this dynamic programming framework because of the short planning horizon in practical applications. The set of feasible acquisition decisions \( X_t \) is defined by the following constraints corresponding to the budget and availability limitations, respectively:

\[
c_t x_t \leq B_t \quad t = 1, \ldots, T - 1
\]

\[
x_t \leq Q_t \quad t = 1, \ldots, T - 1
\]

### 2.2 The Multistage Stochastic Programming Model

In this section, we present a multistage stochastic programming formulation for the housing acquisition problem. Note that a more detailed presentation of this model is given by [1]. This formulation is used mostly for advanced numerical analysis.

Let \( \omega \in \Omega \) represent a scenario corresponding to a possible realization of the stochastic parameters in the problem, i.e. \( \varepsilon_{\mu}, \varepsilon_{Q} \). If \( p^{\omega} \) is the probability of scenario \( \omega \), then the following stochastic programming formulation can be defined for the foreclosed housing acquisition problem:

\[
\max z = \sum_{\omega \in \Omega} p^{\omega} \left( \sum_{t=1}^{T} \sum_{i=1}^{N} (\mu_t^{\omega} - c_t^i) x_{t,i,\omega} + \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{i'=1}^{N} x_{t,i,\omega} x_{t',i',\omega} \right)
\]
\[ \sum_{i=1}^{N} c_i^i x_{i,0}^i \leq B_t \quad \forall t, \omega \quad (6) \]

\[ x_{i,0}^i \leq Q_{i,0}^i \quad \forall i, \omega \quad (7) \]

\[ \sum_{t'=1}^{t} x_{i,t'} \leq Q_{i,0}^i + \sum_{t'=1}^{t} \epsilon_{Q_{i,t'}} + M_t (1 - \nu_{t,0}^i) \quad \forall i, t \neq 1, \omega \quad (8) \]

\[ x_{i,0}^i \leq M_t \nu_{t,0}^i \quad \forall i, t \neq 1, \omega \quad (9) \]

\[ \sum_{t'=1}^{t} x_{i,t'} + (Q_{i,0}^i + \sum_{t'=1}^{t} \epsilon_{Q_{i,t'}}) \nu_{t,0}^i \geq Q_{i,0}^i + \sum_{t'=1}^{t} \epsilon_{Q_{i,t'}} + 1 \quad \forall i, t \neq 1, \omega \quad (10) \]

Nonanticipativity constraints \( \forall i, t \neq T, \quad (11) \)

\[ x_{i,t} \geq 0 \quad \forall t, \omega \quad (12) \]

\[ \nu_{t,0}^i \in \{0, 1\} \quad \forall t \neq 1, \omega \quad (13) \]

where \( M_t \) is a practical upperbound for \( \sum_{t'=1}^{t} x_{i,t'} \), and \( \nu_{t,0}^i \) is a binary variable such that \( \nu_{t,0}^i \) is equal to 1 if \( (Q_{i,0}^i + \sum_{t'=1}^{t} \epsilon_{Q_{i,t'}} - \sum_{t'=1}^{t} x_{i,t'}^i) \geq 0 \), and 0 otherwise. In other words, \( \nu_{t,0}^i \) is used to ensure that the availability constraint always has a non-negative right hand side. Equation (5) represents our objective function, defined similarly to (1). Constraint (6) is the budget constraint for all periods, while constraint (7) is the first stage availability constraint. In addition, constraints (8) and (9) ensure that the number of acquired units in periods \( t > 1 \) is less than the number of remaining housing units. Constraints (8) and (10) together are used to define the binary variable \( \nu_{t,0}^i \). Constraints (11) are the nonanticipativity constraints for this multistage problem, which impose the condition that scenarios that share the same history until a decision epoch also make the same decisions during that history. For the sake of briefness, we omit explicit descriptions of these constraints, but they are modeled through the standard implementation of setting nonanticipative scenario variables equal to each other at each stage. Constraints (12) and (13) ensure the nonnegativity and binariness of the corresponding decision variables.

### 3 Optimal Policy Characterization for Two Special Cases

In this section, we analyze the optimal investment strategy for two special cases of the foreclosed housing acquisition problem. In special case (1) we assume a 2 period problem with 2 housing categories. We further assume that these two categories differ only in the variances of their second period return. Without loss of generality we assume that \( Q_1^i = Q_2^i = Q, c_1^i = c_2^i = c, \) and \( B_1 = B_2 = B \). It is further assumed that, \( \epsilon_Q \) is deterministic and known.

We state the optimal acquisition policy for this special case as follows:

**Proposition 1** For special case 1, the following policy is optimal for first period investment decisions. If \( 2cQ \leq B \), then \( x_{i,0}^* = Q \). If \( 2cQ > B, \) and \( \frac{2B}{c} > Q + \epsilon_Q \geq \frac{B}{c} \); then
\[ Q + \varepsilon_Q - \frac{B}{c} \leq x_1^{1*} \leq 2 \frac{B}{c} - Q - \varepsilon_Q \quad \text{and} \quad x_2^{1*} = \frac{B - x_1^{1*}}{c} \]

If \( 2cQ > B, \) and \( Q + \varepsilon_Q < \frac{B}{c} \) or \( Q + \varepsilon_Q > \frac{2B}{c} \); then \( x_1^{1*} = \frac{B}{c} \) and \( x_2^{1*} = 0 \)

**Proof:** Proof is included in the Appendix.

The optimal policy implies that if there is sufficient budget, all of the units should be acquired in any given period. If this is not the case the optimal policy depends on the values of \( Q, \varepsilon_Q \) and \( B \). If \( \frac{2B}{c} > Q + \varepsilon_Q \geq \frac{B}{c} \), the objective function will be concave and there will be multiple solutions. If \( Q + \varepsilon_Q < \frac{B}{c} \) and \( Q + \varepsilon_Q \frac{B}{c} \), the objective function will be convex and the optimal solutions will be to invest in one category only.

For special case (2) we consider the same problem as in special case (1) but for different \( \mu_i \) values for \( i=1,2 \). The optimal policy is for this case is stated as follows:

**Proposition 2** For special case (2), the following policy is optimal for first period investment decisions. If \( 2cQ \leq B \), then \( x_i^{1*} = Q \). If \( 2cQ > B \) and \( \mu_1 > \mu_2 \); then

\[ x_1^{1*} = \min\{Q, \frac{B}{c}\} \quad \text{and} \quad x_2^{1*} = 0 \]

**Proof:** Proof is included in the Appendix.

This policy implies that if there is sufficient budget, all of the units should be acquired in any given period. If this is not the case; a greedy approach is optimal and as many units as possible should be acquired from the category with the highest marginal return. Any remaining resources should be used to acquire as many units as possible from the other category.

### 4 Conclusions and Future Work

In this paper we study stochastic dynamic models for foreclosed housing acquisition and redevelopment. To this end we first present a dynamic programming formulation for two special cases and identify the optimal policies. More specifically, we extend previous results in the literature, i.e. in [1], to include models without the restrictive assumption of dominance relationship. For special case (2), we show that the optimal investment policy is a greedy policy based on the marginal return values of each category, when \( \mu_1 \neq \mu_2 \).

In this study, we have assumed that market conditions are stable with respect to the distribution of the return values. An extension of our model is to consider more comprehensive cases where the expected change in utilities is non-zero, reflecting increasing or decreasing market conditions. From a practical perspective, future research should also integrate model results with soft operations research methods to create practical and implementable decision rules. Furthermore, real data should be used to convert multiple criteria into a utility based single objective.
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Appendix: Proofs

Proof of Proposition 1

Proof Consider the 2-period 2-category problem. By only considering period 2, it is clear that the optimal solution for this one period knapsack problem is as follows:

\[ x_2^1 = \min \{ Q, \frac{B}{c} \} \text{ and } x_2^2 = \frac{B - \min \{ Q, \frac{B}{c} \}}{c} \text{ or } x_2^2 = \frac{B - \min \{ Q, \frac{B}{c} \}}{c} \text{ and } x_2^2 = \min \{ Q, \frac{B}{c} \} \]

Given this, it follows that:

\[
V_1(\mu_1, \Phi_1, Q_1) = \max_{x_t \in \mathcal{X}_1} \left\{ \sum_{i=1}^{2} \mu_i^t \Phi_i^t + (\mu_i^t - c_i^t)x_t^i + E_{\epsilon_i^t, \mu_i} \left[ \sum_{i=1}^{2} (\mu_i^t + \epsilon_i^t)(\Phi_i^t + x_t^i) + \max \{ (\mu_i^t + \epsilon_i^t - c_i^t) \min [Q_i^t + \epsilon_i^t - x_t^i, \frac{B_i^t}{c_i^t}], (\mu_i^t + \epsilon_i^t - c_i^t) \min [Q_i^t + \epsilon_i^t - \frac{B_i^t}{c_i^t}], \frac{B_i^t}{c_i^t} \} \right] \right\}
\]

\[
(14)
\]

Without loss of generality, assume that \( \Phi_1 = \Phi_2 = 0 \). Under special case 1, it is also assumed that \( Q_1 = Q_2 = Q \), \( c_1 = c_2 = c \) for all \( t \), \( \mu_1 = \mu_2 = \mu \), and \( B_1 = B_2 = B \). Given that \( E[\epsilon_{\text{st}}] = 0 \),

\[
V_1(\mu_1, \Phi_1, Q_1) = \max_{x_t \in \mathcal{X}_1} \left\{ (2\mu - c)x_t^1 + (2\mu - c)x_t^2 + E[\max \{ B(\frac{\mu}{c} + \frac{\epsilon_i}{c} - 1) + \min [Q + \epsilon_i - x_t^1, \frac{B}{c}[\epsilon_i - \epsilon_i^t]], B(\frac{\mu}{c} + \frac{\epsilon_i}{c} - 1) \} \right\}
\]

\[
(15)
\]

In the above relationship, the value of the inner max function is determined according to the sign of \( \Delta = \epsilon_{\mu_1} - \epsilon_{\mu_2} \). For example, if \( \Delta \geq 0 \), then:

\[
\min [Q + \epsilon Q - x_t^1, \frac{B}{c}] \geq \frac{B}{c} - \min [Q + \epsilon Q - \frac{B}{c} + x_t^1, \frac{B}{c}],
\]

\[
(16)
\]

(17)
while the opposite holds for $\Delta \leq 0$. In the former case, the change in $R = \min[Q + \varepsilon x_1^t - x_1^t, \frac{B}{\varepsilon}]$ will characterize the change in the value of the objective function, which is non increasing in $x_1^t$. When $\Delta \leq 0$, the change in the function of $S = \min[Q + \varepsilon x_1^t - \frac{B}{\varepsilon} + x_1^t, \frac{B}{\varepsilon}]$ will characterize the change in the value of the objective function. In this case the objective function is nondecreasing in $x_1^t$. Thus, the expectation based on the probabilities that $\Delta \geq 0$ and $\Delta < 0$ would imply either a convex function or a piecewise concave function as shown in Figure 1. Thus, the optimal policy can be described as follows:

![Figure 1: Optimal Acquisition Policy](image)

If $2cQ \leq B$, then $x_i^t = Q$. If $2cQ > B$, and $\frac{2B}{\varepsilon} > Q + \varepsilon_Q > \frac{B}{\varepsilon}$; then

$$Q + \varepsilon_Q - \frac{B}{\varepsilon} \leq x_1^t \leq 2\frac{B}{\varepsilon} - Q - \varepsilon_Q \quad \text{and} \quad x_1^t = \frac{B - x_1^t + \varepsilon}{\varepsilon}$$

If $2cQ > B$, and $Q + \varepsilon_Q < \frac{B}{\varepsilon}$ or $Q + \varepsilon_Q > \frac{2B}{\varepsilon}$; then

$$x_1^t = \frac{B}{\varepsilon} \quad \text{and} \quad x_1^t = 0$$

**Proof of Proposition 2**

**Proof** In the proof of the Proposition 1 we show how the changes in the values of $R$ and $S$ affect the optimal value. But in special case (2), since $\mu_1^t \neq \mu_2^t$, the value of the objective function will also be affected by marginal return values. The effect of the marginal returns is linear. For instance, if $\mu_1^t < \mu_2^t$, as $x_1^t$ increases by one unit, the objective function value will decrease by $2(\mu_2^t - \mu_1^t)$ units, i.e. it is nonincreasing in $x_1^t$. So in order to figure out the optimal solution we add this linear effect to $R$ and $S$ functions separately. The general representation of the above effects is described in Figure 2.

Based on this definition, optimal policy can be observed to be greedy policy, and can be described as follows: If $2cQ \leq B$, then $x_i^t = Q$.

If $2cQ > B$ and $\mu_1^t > \mu_2^t$; then
Figure 2: Optimal Acquisition Policy for different marginal returns

\[ x_1^* = \min \left\{ Q, \frac{R}{c} \right\} \quad \text{and} \quad x_2^* = 0 \]

References


