Inventory models for deteriorating items with discounted selling price and stock dependent demand

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Abstract
New families of inventory models are developed with pre- and post-deterioration discounts when demand is initially stock dependent and then becomes constant. Shortages are allowed and partially backordered depending on the waiting time for the next replenishment. Furthermore, the time value of money is considered over a fixed planning horizon.

Keywords: Inventory, Deteriorating, Inflation

Introduction
Deteriorating inventory models accounting for partial back-ordering and financial aspects such as pricing strategies, inflation and time value of money are well covered in the current literature; however, there are some characteristics in the modeling of inventory systems with deteriorating goods hardly studied. For instance, incorporating a known price discount is very common and corresponds very well with the real world situations for deteriorating items, but a few authors consider two deterioration phases associated to two subsequent price discounts.

prices dynamically for items with fixed lifetime, but shortages are not considered. Dye and Hsieh (2011) developed an inventory model under inflation for deteriorating items with price- and stock-dependent demand and limited capacity. Meanwhile, Valliathal and Uthayakumar (2011) presented a production lot-sizing model for deteriorating item under inflation by considering a coordinated pricing and production decisions as well as price decision disregarding the production cost. The major assumptions used in the existing works with pricing decision and partially backordered depending on the waiting time have been summarized in Table 1.

Table 1 - Summary of related literature for inventory models with partially backlogging

<table>
<thead>
<tr>
<th>Author(s) and year</th>
<th>Deterioration</th>
<th>Demand</th>
<th>Time value of money</th>
<th>Time horizon</th>
<th>Pricing decision per inventory cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Abad 2003)</td>
<td>Constant (instantaneously)</td>
<td>Price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Ghosh et al. 2011)</td>
<td>Constant (instantaneously)</td>
<td>Price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Dye 2012)</td>
<td>Time dependent (instantaneously)</td>
<td>Time and price dependent</td>
<td>No</td>
<td>Finite</td>
<td>Two-phase pricing</td>
</tr>
<tr>
<td>(Maihami and Abadi 2012)</td>
<td>Constant (non-instantaneously)</td>
<td>Time and price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Maihami and Nakhai Kamalabadi 2012)</td>
<td>Constant (non-instantaneously)</td>
<td>Time and price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Valliathal and Uthayakumar 2010)</td>
<td>Constant (non-instantaneously)</td>
<td>Time and price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Dye and Hsieh 2011)</td>
<td>Constant (instantaneously)</td>
<td>Stock level and price dependent</td>
<td>Yes</td>
<td>Finite</td>
<td>Dynamic pricing</td>
</tr>
<tr>
<td>(Dye and Hsieh 2013)</td>
<td>Constant (instantaneously)</td>
<td>Price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Two-phase pricing</td>
</tr>
<tr>
<td>(Soni and Patel 2012)</td>
<td>Constant (non-instantaneously)</td>
<td>Price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Shavandi et al. 2012)</td>
<td>Constant (instantaneously)</td>
<td>Prices dependent</td>
<td>No</td>
<td>Finite</td>
<td>One pricing decision Per item</td>
</tr>
<tr>
<td>(Teng et al. 2007)</td>
<td>Constant (instantaneously)</td>
<td>Price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Abad 2001)</td>
<td>Time dependent (instantaneously)</td>
<td>Price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Dye et al. 2007)</td>
<td>Time dependent (instantaneously)</td>
<td>Price dependent</td>
<td>No</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>(Valliathal and Uthayakumar 2011)</td>
<td>Time dependent (instantaneously)</td>
<td>Price and time dependent</td>
<td>Yes</td>
<td>Infinite</td>
<td>Equals</td>
</tr>
<tr>
<td>Present paper</td>
<td>Constant (non-instantaneously)</td>
<td>Stock dependent and constant</td>
<td>Yes</td>
<td>Finite</td>
<td>Two-phase price reduction</td>
</tr>
</tbody>
</table>

As a consequence, according to the present authors’ knowledge, no one has attempted to introduce a temporary price reduction on selling price before the start of deterioration as well as a discount on selling price as the deterioration starts in order to boost the inventory depletion rate when inventory systems are dealing with both waiting time dependent backlog rate and deteriorating inventory accounting for inflation and time value of money.
So, the current paper represents above issue in detail extending the model in Panda et al. (2009). In what remains, this paper is organized in the following order. First, it formulates the basic mathematical model with pre- and post-deterioration discounts on selling price. Second, the special cases from the formulation of the basic model are submitted. Third, a numerical example is presented to exemplify the model developed. And finally, the conclusion of this study is given.

**Mathematical Model**

In this paper, a deterministic inventory model with pre- and post-deterioration discounts on selling price in order to investigate how much discount on selling price may be given to maximize the profit over a finite horizon $H$ is considered. The total time horizon $H$ has been divided into $m$ equal parts of length $T_B$ (i.e., $T_B = H/m$). At the beginning of the replenishment cycle the inventory level raises to $Q_1$. As time progresses it decreases due to instantaneous stock dependent demand up to the time $\tau$. After $\tau$ deterioration starts and the inventory level decreases for deterioration and constant demand until it reaches zero level at $T_1$. Ultimately, backorders are accumulated from time $T_1$ to $T_B$. So, the evolution of inventory level occurs according to the following system of linear differential equations

\[
\begin{align*}
\frac{dl(t)}{dt} & = -a - bl(t), & 0 \leq t \leq t_1 \\
& = -a_1 (a + bl(t)), & t_1 \leq t \leq \tau \\
& = -a_2 a - \theta l(t), & \tau \leq t \leq T_1 \\
& = -e^{-\beta(T_B-t)}a, & T_1 \leq t \leq T_B
\end{align*}
\]

Where $a > 0$ is the initial demand rate independent of stock level, $b > 0$ is the stock sensitive demand parameter, $l(t)$ is the instantaneous inventory level at time $t$, $a_1 = (1 - r_1)^{-n_1}$ ($n_1 \in R$), is the effect of $r_1$ discount over demand, $a_2 = (1 - r_2)^{-n_2}$ ($n_2 \in R$), is the effect of $r_2$ discounted selling price over demand, $\theta$ is the deterioration rate $(0 < \theta \ll 1)$, and $e^{-\beta(T_B-t)}$ is the fraction rate of backordered items at time $t$ with $0 < \beta \leq 1$. Note that the fraction of shortages backordered is a decreasing function of time $t$, and it reflects the fact that less waiting time implies more backordered items. Also note that $n_1, n_2$ and $\beta$ are determined from prior knowledge of the seller depending on the influence caused by the reduction rate of selling price and the waiting time over demand rate in each case respectively.

Solving the differential equations with the initial and boundary conditions we get

\[
\begin{align*}
I(t) & = \frac{a}{b} (e^{-bt} - 1) + Q_1 e^{-bt}, & 0 \leq t \leq t_1 \\
I(t) & = \frac{a}{b} (e^{a_1 b(t_1-t)-bt_1} - 1) + Q_1 e^{a_1 b(t_1-t)-bt_1}, & t_1 \leq t \leq \tau \\
I(t) & = \frac{a a_2}{\beta} [e^{\beta(T_1-t)} - 1], & \tau \leq t \leq T_1 \\
I(t) & = \frac{a}{\beta} e^{-\beta(T_B-t)} [e^{-\beta(t-T_1)} - 1] & T_1 \leq t \leq T_B
\end{align*}
\]

Now, at the point $t = \tau$ we have from Equations (6) and (7)
\[
Q_1 = \left( \frac{\alpha a_2}{\theta} e^{\alpha(T_1 - \tau)} - 1 \right) + \frac{\alpha}{b} e^{\alpha_1 b(r - t_1) + bt_1} - \frac{a}{b} \quad (9)
\]

The present value of the holding cost and disposal cost during the entire horizon \(H\) are

\[
HC + DC = \left[ h \int_0^{t_1} e^{-rt} I(t) dt + h \int_{t_1}^\tau e^{-rt} I(t) dt + (h + \theta d) \int_{\tau}^{T_1} e^{-rt} I(t) dt \right] \frac{1 - e^{-rH \frac{H}{m}}}{1 - e^{-rH \frac{H}{m}}} \quad (10)
\]

Where \(h\) and \(d\) are the respective holding and disposal costs per unit and \(m\) denotes the number of replenishment periods during the time horizon \(H\).

The remaining costs during the entire time horizon \(H\) are found as follow:

Let \(c\) the per unit purchase cost of the items. Because shortage during the last cycle is replenished at time \(m \cdot T_B = H\), and shortage during the first replenishment cycle should be backordered during the next replenishment cycle, the present worth of the purchasing cost is given by

\[
PC = c \left\{ \sum_{j=1}^m Q_1 e^{-(j-1)rT_B} + \sum_{j=1}^m [-I(T_B)] e^{-jrT_B} \right\} = c \left[ Q_1 \frac{1 - e^{-rH \frac{H}{m}}}{1 - e^{-rH \frac{H}{m}}} + \frac{a}{\beta} \left( 1 - e^{-\beta \frac{H}{m} - T_1} \right) \frac{1 - e^{-rH \frac{H}{m}}}{1 - e^{-rH \frac{H}{m}}} \right] \quad (11)
\]

Let \(p\) be the shortage cost for backordered items. The present value of the total shortage cost during the entire time horizon \(H\) is equal to

\[
SC = \sum_{j=1}^m \left[ -p \int_{T_1}^{T_B} e^{-rt} I(t) dt \right] e^{-jrT_B}
= -\frac{p \cdot a}{\beta(\beta - r)} \left[ e^{-rT_1 - \beta \frac{H}{m} - T_1} \left( 1 + \frac{\beta - r}{r} \right) - e^{-rH \frac{H}{m}} \left[ 1 + \frac{\beta - r}{r} e^{-\beta \frac{H}{m} - T_1} \right] \right] \frac{1 - e^{-rH \frac{H}{m}}}{e^{H \frac{H}{m} - 1}} \quad (12)
\]

Defining \(l\) as the per unit shortage cost for lost items, the present value of the total shortage cost of the items being lost in the entire time horizon \(H\) is given by

\[
LC = \sum_{j=1}^m \left[ l \cdot a \int_{T_1}^{T_B} e^{-rt} [1 - e^{-\beta(T_B - t)}] dt \right] e^{-jrT_B}
= \frac{l a}{r} \left[ e^{-rT_1} \left( 1 + \frac{r}{\beta - r} e^{-\beta(H/m - T_1)} \right) - e^{-rH/m} \left( 1 + \frac{r}{\beta - r} \right) \right] \frac{1 - e^{-rH \frac{H}{m}}}{e^{H \frac{H}{m} - 1}} \quad (13)
\]

Defining \(s\) as the per unit selling price of the items. The present worth of the total sales revenue in the entire horizon \(H\) can be found by equation (14).
Thus, the net present value of the system during the entire horizon H (including set up cost $C_0$) is

$$Z_1(r_1, r_2, t_1, T_1, m) = SR - PC - HC - DC - C_0 \frac{e^{rH/m} - e^{-rH}}{e^{r/m} - 1} - LC - SC$$

(15)

On integration and simplification of the relevant costs, the net profit value during the entire horizon H becomes (16). Where, PC, SC, LC are equal to equations (11), (12), (13) respectively

$$Z_1(r_1, r_2, t_1, T_1, m) = s \frac{a}{r} \left( e^{-rT_1} \gamma_1 - 1 \right) + e^{-rT_1} \left( \gamma_2 - \gamma_1 \right) + \left( 1 - \gamma_2 e^{-rT_1} \right) \frac{1 - e^{-rH}}{e^{rH/m} - 1}$$

$$+ \left( sb \frac{1 - e^{-rH}}{e^{rH/m} - 1} - h \frac{1 - e^{-rH}}{e^{r/m} - 1} \right) \int_0^{t_1} e^{-rT_1} I(t) dt + \left( s \gamma_1 b \frac{1 - e^{-rH}}{e^{rH/m} - 1} - h \frac{1 - e^{-rH}}{e^{r/m} - 1} \right) \int_{t_1}^T e^{-rT_1} I(t) dt$$

$$- (h + \theta d) \frac{1 - e^{-rH}}{1 - e^{-rH/m}} \int_{t_1}^T e^{-rT_1} I(t) dt - PC - LC - SC - C_0 \frac{e^{rH/m} - e^{-rH}}{e^{rH/m} - 1}$$

(16)

Where:

$$\gamma_1 = (1 - r_1)/(1 - r_1)^n_1; \quad \gamma_2 = (1 - r_2)/(1 - r_2)^n_2$$

$$\int_0^{t_1} e^{-rT_1} I(t) dt = \left( \frac{a}{b} + Q_1 \right) \frac{1}{r + b} \left( 1 - e^{-T_1(r+b)} \right) + \frac{a}{br} \left( e^{-rT_1} - 1 \right)$$

$$\int_{t_1}^T e^{-rT_1} I(t) dt = \left( \frac{a}{b} + Q_1 \right) \frac{1}{r + \alpha_1 b} \left( e^{-T_1(r+b)} - e^{-T_1+\alpha_2 b(t_1-\tau) - bt_i} \right) + \frac{a}{br} \left( e^{-r\tau} - e^{-rT_1} \right)$$

$$\int_{t_1}^{T_1} e^{-rT_1} I(t) dt = \frac{a}{r \theta} \left[ e^{-rT_1} \left( \frac{r e^\theta(T_1-\tau)}{r + \theta} - 1 \right) - e^{-rT_1} \left( \frac{r}{r + \theta} - 1 \right) \right]$$

Now, considering the constraints in which the pre- and post-deterioration discounts on selling price is given in such a way that the discounted selling price is not less than the unit cost of the product, the objective here is to determine the optimal values of $r_1, r_2, t_1, T_1 \geq 0$ and $m \in Z^+$ which maximize the function $Z_1$ constrained by $r_1 < 1 - \frac{c}{s} ; r_2 < 1 - \frac{c}{s}$ and $t_1 < r < T_1 \leq H/m$. Note that it’s very difficult to derive the results analytically. Thus, some of the numerical algorithms for constrained nonlinear optimization must be applied to derive the optimal values of $r_1, r_2, t_1, T_1$, and $m$. There are several methods to cope with constraint optimization problem numerically. But here it’s uses The Differential Evolution Algorithm (Price et al. 2005) to derive the optimal values of the decision variables.
Some special cases

Let $Z_1$ be the above model described. The following situations can be easily deduced from equations (9) and (16) as follow:

- **Model $Z_2$ with only post-deterioration discount on unit selling price** $(t_1 = \tau, r_1 = 0)$

In this case only discount on selling price will be given as soon as the deterioration starts. So, the order quantity is obtained from equation (9) by Substituting $t_1 = \tau$ and $r_1 = 0$ as

$$Q_2 = \left(\frac{a\alpha_2}{\theta} (e^{\theta(T_1 - \tau)} - 1) + \frac{a}{b}\right) e^{br} - \frac{a}{b}$$

(17)

Also, from equation (16) the net present value of the system is found as

$$Z_2(r_2, T_1, m) = s \left[ \left(1 - e^{-rT_1}\right) + \frac{(1 - r_2)}{(1 - r_2)n^2} (e^{-rT_1} - e^{-rT_1}) \right] 1 - \frac{e^{-rH}}{e^{\frac{rH}{m}} - 1} - PC - LC - SC$$

$$+ \left( sb \frac{1 - e^{-rH}}{e^{\frac{rH}{m}} - 1} - h \frac{1 - e^{-rH}}{e^{\frac{rH}{m}} - 1} \right) \left[ \left(\frac{a}{b} + Q_2\right) \left(\frac{1}{r + b}\right) (1 - e^{-\tau(r+b)}) + \frac{a}{br} (e^{-\tau} - 1) \right]$$

$$- (h - \theta d) \left(\frac{1 - e^{-rH}}{1 - e^{-\frac{rH}{m}}} \right) a \theta \left[ e^{-\tau \left(\frac{r e^{\theta(T_1 - \tau)}}{r + \theta} - 1\right)} - e^{-rT_1} \left(\frac{r}{r + \theta} - 1\right) \right] - C_0 \frac{e^{rH/m} - e^{-rH}}{e^{rH/m} - 1}$$

(18)

Therefore, the goal here is to find the optimal values of $r_2, T_1 \geq 0$ and $m \in Z^+$ to maximize the function $Z_2$ constrained by $r_2 < 1 - \frac{c}{s}$ and $\tau < T_1 \leq H/m$. Where $PC, SC, LC$ are equal to equations (11), (12), (13) respectively, bearing in mind that $Q_1$ is $Q_2$ in (11) now.

- **Model $Z_3$ for no discount on unit selling price** $(t_1 = \tau, r_1 = r_2 = 0)$

In this model there is no pre-deterioration as well as no post deterioration discount on unit selling price. So, the order quantity is found substituting $t_1 = \tau, r_1 = r_2 = 0$ from Equation (9) as

$$Q_3 = \left(\frac{a}{b} (e^{\theta(T_1 - \tau)} - 1) + \frac{a}{b}\right) e^{br} - \frac{a}{b}$$

(19)

And from equation (16) the net present value becomes (20). Where $PC, SC, LC$ are equal to equations (11), (12), (13) respectively, taking into account that $Q_1$ is $Q_3$ in (11) now.

$$Z_3(T_1, m) = s \left[ (1 - e^{-rT_1}) + (e^{-rT_1} - e^{-rT_1}) \right] 1 - \frac{e^{-rH}}{e^{\frac{rH}{m}} - 1} - PC - LC - SC$$

$$+ \left( sb \frac{1 - e^{-rH}}{e^{\frac{rH}{m}} - 1} - h \frac{1 - e^{-rH}}{e^{\frac{rH}{m}} - 1} \right) \left[ \left(\frac{a}{b} + Q_3\right) \left(\frac{1}{r + b}\right) (1 - e^{-\tau(r+b)}) + \frac{a}{br} (e^{-\tau} - 1) \right]$$

$$- (h - \theta d) \left(\frac{1 - e^{-rH}}{1 - e^{-\frac{rH}{m}}} \right) a \theta \left[ e^{-\tau \left(\frac{r e^{\theta(T_1 - \tau)}}{r + \theta} - 1\right)} - e^{-rT_1} \left(\frac{r}{r + \theta} - 1\right) \right] - C_0 \frac{e^{rH/m} - e^{-rH}}{e^{rH/m} - 1}$$

(20)
So, the goal in (20) is to find the optimal values of $T_1$, and $m \in Z^+$ to maximize the function $Z_3$ constrained by $0 < T_1 \leq H/m$.

- **Model $Z_4$ for instant deterioration with post deterioration discount** ($\tau = t_1 = r_1 = 0$)

In this case pre-deterioration discount is no option because the items start to deteriorate as soon as it’s received in the stock. So, from equations (9) the order quantity can be found by substituting $\tau = t_1 = r_1 = 0$ as

$$Q_4 = \frac{a \alpha_2}{\theta} (e^{\theta t_1} - 1) \quad (21)$$

Similarly, from equation (16) net present worth becomes (22). Where PC, SC, LC are equal to equations (11), (12), (13) respectively, considering that $Q_1$ is $Q_4$ in (11) now.

$$Z_4(r_2, T_1, m) = s \frac{a}{r} \left( \frac{1 - r_2}{r} \right) \left( 1 - e^{-rT_1} \right) \left[ \frac{1 - e^{-\tau H}}{e^{-\tau H} - 1} \right] - PC - LC - SC - C_0 \frac{e^{rH/m} - e^{-rH}}{e^{rH/m} - 1}
-(h - \theta d) \left[ \frac{1 - e^{-\tau H}}{1 - e^{-rH/m}} \right] \frac{a \alpha_2}{r \theta} \left( \left( r e^{\theta t_1} \right) \left( \frac{r}{r + \theta} - 1 \right) - e^{-rT_1} \left( \frac{r}{r + \theta} - 1 \right) \right) \quad (22)$$

Hence, the aim here is to determine the optimal values of $r_2$, $T_1$ and $m \in Z^+$ which maximize the function $Z_4$ constrained by $r_2 < 1 - \frac{\epsilon}{5}$ and $0 < T_1 \leq H/m$

- **Model $Z_{41}$ with instant deterioration and without discount** ($\tau = t_1 = 0, r_1 = r_2 = 0$)

The initial inventory level can be obtained from equations (21) by substituting $\tau = t_1 = r_1 = r_2 = 0$ as

$$Q_{41} = \frac{a}{\theta} (e^{\theta t_1} - 1) \quad (23)$$

So, from equation (22) the net present value becomes (24). Where PC, SC, LC are equal to equations (11), (12), (13) respectively with $Q_1 = Q_{41}$ in (11). Thus, we have to determine $T_1$ and $m$ from which the function (24) is maximized and meet $0 < T_1 \leq H/m$ and $m \in Z^+$

$$Z_{41}(T_1, m) = s \frac{a}{r} (1 - e^{-rT_1}) \frac{1 - e^{-\tau H}}{e^{-\tau H} - 1} - PC - LC - SC - C_0 \frac{e^{rH/m} - e^{-rH}}{e^{rH/m} - 1}
-(h - \theta d) \left( \frac{1 - e^{-\tau H}}{1 - e^{-rH/m}} \right) \frac{a}{r \theta} \left[ \left( r e^{\theta t_1} \right) \left( \frac{r}{r + \theta} - 1 \right) - e^{-rT_1} \left( \frac{r}{r + \theta} - 1 \right) \right] \quad (24)$$

- **Model $Z_5$ for fixed life time products with pre deterioration discount** ($\tau \rightarrow T_5, r_2 = 0$)

In this scenario there is only pre-deterioration discount on selling price and cycle length must be less than life time of items. So the order quantity (25) and net present worth (26) for fixed life
time products where $0 < t_1 < T_1 < \tau$ can be found from equation (9) and (16), by letting $\tau \to T_1$ and $r_2 = 0$ as follows

$$Q_5 = \frac{a}{b} \left( e^{\alpha b(t_1-t_1)+bt_1} - 1 \right)$$

$$Z_5 (r_1, t_1, T_1, m) = \frac{s}{r} \left[ e^{-rt_1} \left( \frac{1 - r_2}{1 - r_1} \right) - 1 \right] + \left[ 1 - \frac{(1 - r_2)}{1 - r_1} \right] \frac{1 - e^{-rH}}{e^{\frac{rH}{m}} - 1}$$

$$+ \left( \frac{sb}{e^{\frac{rH}{m}} - 1} - h \frac{1 - e^{-rH}}{1 - e^{\frac{rH}{m}}} \right) \left[ \frac{a}{b} + Q_5 \right] \left( \frac{1}{r + b} \right) \left( 1 - e^{-t_1(r+b)} \right) + \frac{a}{br} (e^{-rt_1} - 1)$$

Thus, the aim is to determine $r_1, t_1, T_1 \geq 0$ and $m \in Z^+$ to maximize (26) constrained by $r_1 < 1 - \frac{c_s}{s}, 0 < t_1 < T_1 < \tau < H/m$, Where PC, SC, LC are equal to (11),(12),(13) respectively, bearing in mind that $Q_1$ is $Q_5$ in (11) now.

- **Model $Z_{51}$ for fixed life time products with no discount** ($\tau, t_1 \to T_1$ and $r_1 = r_2 = 0$)

Letting $t_1 \to T_1$ and $r_1 \to 0$, the initial inventory level and the net present value becomes (27) and (28) from (25) and (26) respectively. So, the objective here is to found $T_1 \geq 0$ and $m \in Z^+$ to maximize (28) constrained by $0 < t_1 < T_1 < \tau$. Where PC, SC, LC are equals to (11),(12),(13) respectively, taking in account that $Q_1$ is $Q_{51}$ in (11) now.

$$Q_{51} = \frac{a}{b} \left( e^{bT_1} - 1 \right)$$

$$Z_{51} (T_1, m) = \frac{s}{r} \left[ \frac{1 - e^{-rT_1}}{e^{\frac{rH}{m}} - 1} - h \frac{1 - e^{-rH}}{1 - e^{\frac{rH}{m}}} \right] \left[ \frac{a}{b} + Q_{51} \right] \left( \frac{1}{r + b} \right) \left( 1 - e^{-T_1(r+b)} \right) + \frac{a}{br} (e^{-rT_1} - 1)$$

**Numerical example**

In order to illustrate the proposed model, an example has been solved with the following parameter values: $a = 90; b = 0.6; s = 10; c = 4.1; h = 0.5; \tau = 1.3; n_1 = n_2 = 1.9; d = 3; C_0 = 180; \theta = 0.03; r = 11%; l = 11; p = 5; \beta = 0.80; H = 7$. As was mentioned it used The Differential Evolution Algorithm (Price et al. 2005) to derive the optimal values of the decision variables. For model with both pre- and post-deterioration discounts ($Z_1$), the pre-deterioration discount on unit selling price is 0.59 ($r_1$) and discount starts at time $t_1 = 0.857$ and continued to $\tau = 1.3$.Then the product start to deteriorates. During this time in order to boost inventory depletion rate, 59% discount is provided for remaining time of replenishment cycle ($r_2$). The net present worth is 3994 ($z_1$). The optimal order quantity is 2494 ($Q_1$) and the cycle length is 1.75($T_B$). These results and the others subcases ( $Z_2$ to $Z_{51}$) are listed in Table 1.
As can be noted, the net present value in model $Z_1$ is greater than model $Z_2$ and model $Z_3$, but there are combinations in the parameters where it can be different. So, a more deep analysis is needed to know when it is useful for managers each models in order to maximize the profit when the time value of money is taken into consideration.

<table>
<thead>
<tr>
<th>Models</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$t_1$</th>
<th>$T_1$</th>
<th>$T_B = H/m$</th>
<th>$Q$</th>
<th>$Z$</th>
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<tbody>
<tr>
<td>$Z_1$</td>
<td>0.59</td>
<td>0.59</td>
<td>0.856602</td>
<td>1.75</td>
<td>1.75</td>
<td>2494.22</td>
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<tr>
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<td>-</td>
<td>-</td>
<td>0.777778</td>
<td>75.4906</td>
<td>3615.75</td>
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<tr>
<td>$Z_3$</td>
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<td>-</td>
<td>-</td>
<td>1.75</td>
<td>1.75</td>
<td>266.17</td>
<td>1973.32</td>
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<td>$Z_4$</td>
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<td>0.59</td>
<td>-</td>
<td>4.56344</td>
<td>-</td>
<td>2395.04</td>
<td>168 828</td>
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<tr>
<td>$Z_{41}$</td>
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<td>-</td>
<td>0.7</td>
<td>4070.68</td>
<td>177.221</td>
<td>1569.13</td>
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<tr>
<td>$Z_5$</td>
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<td>-</td>
<td>0.340424</td>
<td>1.3</td>
<td>1.4</td>
<td>4070.68</td>
<td>828 408</td>
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<tr>
<td>$Z_{51}$</td>
<td>-</td>
<td>-</td>
<td>1.3</td>
<td>1.4</td>
<td>177.221</td>
<td>1569.13</td>
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</table>

**Conclusion**

Panda et al. (2009) presented a practical inventory model with pre- and post-deterioration discounts on selling price. The mathematical model is developed in order to investigate how much discount on selling price may be given to maximize the profit per unit time when demand is both stock dependent and constant, but the time value of money and shortage are not considered. In this paper, that model has been extended to make it a little more applicable when inventory systems are dealing with the inventory replenishment problem for partial backlogging and time value of money.

Although this research represent an important contribution of existing inventory models for deteriorating items with temporary price discounts, the model developed here can be further improved by employing more elaborate inventory models. Potential topics for notable future research include the incorporation of stochastic demand rate, multi-echelon inventory and multi-item approaches.

**References**


