Economic Decision Making Using Fuzzy Numbers
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Abstract

In engineering economic studies, single values are traditionally used to estimate the cash flows. Since uncertainty exists in estimating cost data, the resulting decision may not be reliable. To overcome the shortcoming of single-valued estimation, the fuzzy numbers could be applied in cash flow analysis. Instead of single-valued estimation, each cost data could be designated as a trapezoidal fuzzy set in this approach. The final result from the cash flow analysis will still be a trapezoidal fuzzy set, which provides the decision-maker with a broader view of possible outcomes. Since the final results are trapezoidal fuzzy number sets, comparison among alternatives is not as straightforward as the traditional economic analysis. A ranking method is recommended in this research to assist decision maker in selecting the best alternative.

I. Introduction

Cash flow analysis is essential for sound decision making in engineering economic studies. In traditional cash flow analysis, Present Worth, Annual Cash Flow, Internal Rate of Return, and Benefit-Cost Ratio are the most widely used approaches to perform the economic analysis. If sufficient cost data are available, the traditional approaches are well capable of reaching decisions. Unfortunately, decision makers rarely have enough information to perform the economic analysis. Moreover, because the traditional economic analysis uses single-valued estimates, a small change in cost data may cause completely reversed decision. A good alternative may have been ignored due to a minor error in estimating cost data.

In order to solve the problem of inflexibility and incompleteness of using single-valued estimates for cost data, the fuzzy set concept can be employed to deal with the uncertainty in the cash flow analysis. In real life, decisions sometimes have to be made under the context of incomplete knowledge. It is very likely that decision makers give assessments based on their knowledge, experience, and subjective judgment when estimating cost data. If the cost data are not completely known, linguistic terms such as “around 10 years,” “approximately 30,000 to 45,000 dollars,” “about 9%,” are frequently used to make estimations. Fuzzy set theory can play a significant role in decision making with uncertainty and vagueness because it was oriented for the rationality of uncertainty due to imprecision or vagueness.

Although it was invented in the 1920's, the fuzzy set theory was formally introduced by Zadeh [5] in 1965. The fuzzy set theory is primarily concerned with quantifying the vagueness in human thoughts and perceptions. Linguistic terms could be properly represented by the approximate reasoning of fuzzy set theory [4]. The theory also allows mathematical operators such as addition, subtraction, multiplication, and division, to be applied to the fuzzy domain [4] [2]. Hence, the fuzzy number can be employed in economic analysis to replace the single-valued estimation for vague cost data.
Fuzzy number has been successfully employed in Present Worth Criterion by using triangular fuzzy number [1]. Instead of using triangular number, this research focused on trapezoidal fuzzy numbers to characterize fuzzy measures of linguistic values. The reason for using the trapezoidal fuzzy number is that it is more representative to linguistic estimations in economic analysis [3]. For example, an expert frequently indicates that the first investment is most likely to be between 20 to 22 million dollars, but it can be as low as 18, or as high as 24. In this situation, the first investment could be denoted by (18, 20, 22, 24). In fact, a triangular fuzzy number is a special case of a trapezoidal fuzzy number. When the two most promising values are the same number, the trapezoidal fuzzy number becomes a triangular fuzzy number. Hence, a trapezoidal fuzzy number can deal with more general situations.

The overall objective of this research is to develop a methodology to incorporate the fuzzy number in cash flow analysis. Instead of single-valued estimation used in traditional analysis, fuzzy numbers can quantify the vagueness in human thoughts and perceptions as a range of possible values. In this research, each cost data can be stated as a fuzzy set $\tilde{P}(a, b, c, d)$ where $a$ is the smallest possible value, $b$ and $c$ define the range of the most promising values, and $d$ is the largest possible value. The methodology of applying the traditional cash flow analyses with fuzzy numbers will also be developed.

Instead of incorporating the traditional operation rules, a set of modified operation rule is proposed to deal with cash flow analysis. Although the traditional operation rules could be applied in many areas, such as in electrical control area, they are not suitable in the engineering economic domain. The modified operation rules are developed in conformance with other engineering economic formula. In addition, the resulting range expansion with the modified operation rules is more reasonable.

2. Research Methodology

In this research, the trapezoidal fuzzy sets have been incorporated successfully for four commonly used criteria, namely NPW, EUAW, IRR, and B/C, in cash flow analysis. In order to help decision makers to rank the alternatives, the procedure of a new ranking method is also developed in the paper. Before using the trapezoidal fuzzy method to handle cash flow analysis, the sensitivity analysis is presented to illustrate the influence of each cost data after fuzzy number operations.

In order to prove that the trapezoidal fuzzy method is suitable in cash flow analysis, the trapezoidal fuzzy numbers will be incorporated in the following four criteria: namely $\tilde{NPW}$, $\tilde{EUAW}$, $\tilde{IRR}$, and $\tilde{B/C}$. Like the traditional decision making, the preference should be the same no matter which criterion is used. Thus, the results from the four criteria will be compared for consistency.

2.1 Present Worth Approach

In the criterion of $\tilde{NPW}$, the results of $\tilde{NPW}$ can be obtained by the traditional NPW formula and fuzzy operation rules. Because the final fuzzy set is not obvious to be compared among alternatives, the recommended ranking procedure can be incorporated to make the final decision. The preference with highest value of $\tilde{NPW}$ can be selected as the best alternative.
In the situation of a single alternative, the acceptance can be decided by comparing the final fuzzy set with zero. If the final fuzzy set is greater than zero, the project is desirable. Otherwise, it can be discarded. Sometimes it is not obvious to judge if the final fuzzy set is greater than zero or not by the ranking method. To deal with the situation, there are two approaches to make the decision. The first approach is that one can try different proposed ranking methods for references and then make decision. The second method is to reconsider the fuzzy set for each cost data.

For multiple alternatives, whether independent or mutually exclusive alternatives, the preferences can be established by comparing the value of $NP\tilde{W}$. Since the comparison must be based on the same useful life under the criteria of NPW, the method of $NP\tilde{W}$ becomes very complicated for alternatives with different useful lives. Thus, if the analysis periods of alternatives are different, it is recommended that the criterion of $EUAW\tilde{W}$ is more suitable to deal with the cash flow analysis.

### 2.2 Equivalent Uniform Annual Worth Approach

The criterion of Equivalent Uniform Annual Worth ($EUAW\tilde{W}$) is similar to the criterion of $NP\tilde{W}$ in cash flow analysis. After defining the fuzzy set for each cost data, the traditional EUAW formula and fuzzy operation rules can be applied to calculate the value of $EUAW\tilde{W}$. Since the comparison among alternatives with different useful lives can be made by the criterion of $EUAW\tilde{W}$, the criterion is more flexible than $NP\tilde{W}$. There are two ways of obtaining the results of $EUAW\tilde{W}$. The first procedure is that each cost data is converted to $NP\tilde{W}$ individually and converted to $EUAW\tilde{W}$ together in the first approach. In the second method, each cost data is converted to $EUAW\tilde{W}$ individually and then calculate the summation of these value. Since the formula and the fuzzy operation rules are the same for individual or total value, the final results of $EUAW\tilde{W}$ between the two methods are the same. Thus, decision makers can select each one of these two methods to calculate the $EUAW\tilde{W}$. Based on the values of $EUAW\tilde{W}$, the recommended ranking procedure can be applied to choose the best project with the highest value of $EUAW\tilde{W}$.

### 2.3 Internal Rate of Return Approach

The criterion of Internal Rate of Return ($IR\tilde{R}$) is another approach to deal with cash flow analysis. With trial and error method, $IR\tilde{R}$ can be obtained by satisfying $NP\tilde{W}$ or $EUAW\tilde{W}$ equal to zero. If each number of $IR\tilde{R}$ is greater than MARR, the alternative is desirable. If it is not obvious whether the value of $IR\tilde{R}$ is greater than that of MARR, a weighted method [5] is suggested to help making comparison.

If there are several alternatives with $IR\tilde{R}$ greater than MARR, the incremental rate of return ($ΔRO\tilde{R}$) must be applied to establish the preference. Similar to traditional ΔROR, it is necessary to rank the alternatives according to their first costs. If it is not obvious which alternative has higher first cost, the weighted method [5] can be applied to rank the alternatives. After deciding the sequence of first cost, the incremental rate of return, $ΔRO\tilde{R}$ can be obtained by setting $ΔNP\tilde{W}$ or $ΔEUAW\tilde{W}$ equal to zero.

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2.4 Benefit to Cost Ratio Approach

Similar to traditional Benefit to Cost Ratio (B/C) criterion, the value of \( \frac{\tilde{B}}{\tilde{C}} \) can be calculated if \( EUA\tilde{B} \) is divided by \( EUA\tilde{C} \) or \( NP\tilde{B} \) divided by \( NP\tilde{C} \). If the ratio of \( \frac{\tilde{B}}{\tilde{C}} \) is less than one, the project can be discarded. If there are several alternatives with \( \frac{\tilde{B}}{\tilde{C}} \)-ratios greater than or equal to 1, the incremental \( \frac{\Delta \tilde{B}}{\Delta \tilde{C}} \)-ratio \( \frac{\Delta \tilde{B}}{\Delta \tilde{C}} \) must be used to select the best alternative. The weighted method is suitable to choose the higher first cost and compare the ratio of \( \frac{\Delta \tilde{B}}{\Delta \tilde{C}} \) with 1.

3. Ranking Method

From the above discussion, it is obvious that an appropriate ranking method is very important to choose the best alternative by using trapezoidal fuzzy method. A good ranking method should at least satisfy three criteria, namely consistency, risk reduction, and easy to implement. Though there are many ranking methods proposed, they are not necessarily suitable for economic decision making.

There are two ranking methods applied in this research. In ranking the higher first cost or comparing the \( IR\tilde{R} \) and \( RO\tilde{R} \) with MARR or \( \frac{\tilde{B}}{\tilde{C}} \) and \( \frac{\Delta \tilde{B}}{\Delta \tilde{C}} \) with 1, the weighted method is suggested. The reason is that the effect of fuzzy range is not very significant for those situations. However, the dominate theory and minimax regret methods are applied in comparing the trapezoidal fuzzy sets, such as \( NP\tilde{W} \) and \( EUA\tilde{W} \). When a fuzzy set is clearly superior to the others, the dominate theory can choose the preference quickly. If the dominate theory fails to rank the sequence, the minimax regret criterion is used for decision making. The major contribution of the criterion is that it doesn't focus only on the extreme values. The method takes each number of a fuzzy set into consideration and makes the most conservative decision.

4. Analysis Procedure

The common analysis procedure can be concluded more clearly by the following steps:

Step 1: Identify Cost Data

Before analysis, it is very important to identify cost data for each alternative. Considering the following figure, it is a typical cash flow diagram of an investment project:

![Fig. 1 Cost Data Diagram of a Typical Project](image)

By the traditional analysis, the Equivalent Uniform Annual Worth (EUAW) can be obtained by the following equations:
NPW = \(-F_0 + (F_1 - M_1)(P/F, i, 1) + (F_2 - M_2)(P/F, i, 2) + ... + (F_n + S - M_n)(P/F, i, n)\)
EUAW = \(-F_0 (P/F, i, 0) (A/P, i, n) + (F_1 - M_1) (P/F, i, 1) (A/P, i, n) + (F_2 - M_2) (P/F, i, 2) (A/P, i, n) + ... + (F_n + S - M_n) (P/F, i, n) (A/P, i, n)\)
or
EUAW = NPW(A/P, i, n)

where:

- \(F_0\) = initial investment
- \(i\) = MARR
- \(F_n\) = future revenue in year \(n\)
- \(n\) = useful life
- \(S\) = salvage value
- \(NPW\) = net present worth
- \(EUAW\) = equivalent uniform annual worth
- \((P/F, i, n)\) = present worth factor for single payment = \(1 \div (1+i)^n\)
- \((A/P, i, n)\) = capital recovery factor = \([i(1+i)^n \div ((1+i)^n-1)]\)
- \((A/F, i, n)\) = sinking fund factor = \(i \div [(1+i)^n-1]\)
- \((P/A, i, n)\) = present worth factor for uniform payment series = \([(1 + i)^n - 1] \div [i (1+i)^n]\)

The IRR can be derived by equating either NPW or EUAW to 0.

**Step 2: Evaluate Cost Data**

By the fuzzy analysis method, each cost data related to the cash flow analysis can be designated as a single value estimate or as a fuzzy number. For example, the initial investment can be denoted as \(F_0(a, b, c, d)\), designated as \(\tilde{F}_0\), where \(a\) indicates the smallest possible value (membership value is 0), \(b\) and \(c\) represent the most promising values (membership value is 1), and \(d\) the largest possible value (membership value is 0).

By the same token, the fuzzy set of cost data can be represented as follows:

- \(\tilde{F}_0\) = Fuzzy Initial Investment
- \(\tilde{F}_t\) = Fuzzy Gross Income in year \(t\)
- \(\tilde{E}_t\) = Fuzzy Expenses in year \(t\)
- \(\tilde{S}\) = Fuzzy Salvage Value
- \(\tilde{N}_t\) = Fuzzy Net income in year \(t\)
- \(NP\tilde{W}\) = Fuzzy Net Present Worth
- \(EUA\tilde{W}\) = Fuzzy Equivalent Uniform Annual Worth
- \(IR\tilde{I}\) = Fuzzy Internal Rate of Return
- \(EUAB\tilde{B}\) = Fuzzy Equivalent Uniform Annual Benefit
- \(EUA\tilde{C}\) = Fuzzy Equivalent Uniform Annual Cost

**Step 3: Extend Fuzzy Number Operations**

Based on the fuzzy number and extended fuzzy number operation rule, the following formulas are obtained:

- \(\tilde{N}_t = \tilde{F}_t - \tilde{E}_t\)
- \(NP\tilde{W} = -\tilde{F}_0 + \sum_{t=1}^{n} \left[ \tilde{N}_t \div (1+i)^t \right] + \tilde{S} \div (1+i)^n\)
$EUAW = -\bar{F}_0 (P/F, \bar{i}, 0) (A/P, \bar{i}, \bar{n}) + (\bar{F}_1 - \bar{M}_1) (P/F, \bar{i}, 1) (A/P, \bar{i}, \bar{n})$
$+ \ldots + (\bar{F}_n + \bar{S} - \bar{M}_n) (P/F, \bar{i}, \bar{n}) (A/P, \bar{i}, \bar{n})$

or

$EUAW = NP\bar{W} (A/P, \bar{i}, \bar{n})$

$EUAB = \bar{F}_1 (P/F, \bar{i}, 1) (A/P, i, \bar{n}) + \bar{F}_2 (P/F, \bar{i}, 2) (A/P, i, \bar{n}) + \ldots$
$+ (\bar{F}_n + \bar{S}) (P/F, \bar{i}, \bar{n}) (A/P, i, \bar{n})$

$EUAC = \bar{F}_0 (P/F, \bar{i}, 0) (A/P, i, \bar{n}) + \bar{M}_1 (P/F, \bar{i}, 1) (A/P, i, \bar{n})$
$+ \bar{M}_2 (P/F, \bar{i}, 2) (A/P, i, \bar{n}) + \ldots + \bar{M}_n (P/F, \bar{i}, \bar{n}) (A/P, i, \bar{n})$

$\bar{B} / \bar{C} = EUA\bar{B} / EUA\bar{C}$ or $NP\bar{B} / NP\bar{C}$

By letting $NP\bar{W} = 0$ or $EUAW = 0$, the fuzzy set of Internal Rate of Return, $IR\bar{R}$, can be obtained by trial and error routine.

**Step 4: Ranking**

After the trapezoidal fuzzy number operations, the final fuzzy set on the decision criteria, e.g. $NP\bar{W}$, $EUAW$, $IR\bar{R}$, and $\bar{B} / \bar{C}$ will be obtained for all alternatives. Since those results are fuzzy sets, the selection of a best alternative is not as straightforward as the traditional analysis. Ranking method is a tool that can help the decision makers in selecting the best alternative.

**5. Sensitivity Analysis**

In fuzzy cash flow analysis, the range of the final fuzzy set will serve as the basis for the final decision making. If the range of the final fuzzy set is reasonably small, the preferable alternative can be selected more distinct from the others. On the other hand, if the final fuzzy set covers a wide range after cash flow analysis, the final decision will be very difficult to reach. The purpose of sensitivity analysis is to assess the effects on the final fuzzy set caused by the variation of the cost data. Based on the sensitivity information, decision makers can pay more attention to define the fuzzy set carefully for those sensitive cost data in order to reduce the final range of the fuzzy set.

One common method to reduce the range of a fuzzy set is called $\alpha$-cut, where $\alpha$ is the degree of membership. By properly assigning a value to $\alpha$, the range of the fuzzy set can be reduced. This research will explore the feasibility of using $\alpha$-cut to reduce the range of the fuzzy numbers.

A numerical example will be used to illustrate the concepts of sensitivity analysis and $\alpha$-cut. One must notice that the sensitivity analysis is unique to every single economic problem. The result of the sensitivity analysis can only provide an indication as which cost data is more sensitive in general.

**5.1 Procedure for Sensitivity Analysis**
In general, the procedure is to calculate the NPW or EUAW based on the single-valued estimates. Then, different values of $NPW$ or $EUAW$ can be obtained by changing one cost data into a fuzzy number at a time. By comparing $NPW$ or $EUAW$ with NPW or EUAW, the percentage change can be served as an indicator of sensitivity for the cost data. The cost data which caused most percentage change is defined as the most sensitive. The procedure for sensitivity analysis can be illustrated by a typical engineering economic problem. In the example, the net present worth is used as the criterion in the sensitivity analysis. Five most frequently encountered cost data are incorporated in the sensitivity analysis, namely the first cost ($F_0$), the equivalent uniform annual benefit (EUAB), the minimum acceptable rare of return ($i$), the salvage value ($S$), and the useful life ($n$).

The exemplified problem is first described using single-valued or most likely estimates.

- $F_0 = 1000$ (thousand)
- EUAB = 200 (thousand)
- $i = 10\%$
- $S = 100$ (thousand)
- $n = 20$ years

To show the effects of various cost data on the final fuzzy number, the problem is redefined with fuzzy numbers. In keeping consistency, the fuzzy number of each cost data is composed of -10%, -5%, +5%, +10% of the most likely value. The estimates of cost data are summarized in Table 1.

![Table 1 Cost Data for Sensitivity Analysis](image)

The NPW value is calculated first for the single-valued estimates. The fuzzy cost data is then introduced one at a time to the calculation of $NPW$ by applying the traditional net present worth and fuzzy number operation rules. Since the four numbers for each fuzzy cost data are selected proportional to the most likely value, the percentage difference of $NPW$ for four fuzzy numbers can be calculated by comparing with the value of NPW calculated by traditional method.

### 5.2 Results of Sensitivity Analysis

By the traditional NPW formula and fuzzy number operation rules, the results of the sensitivity analysis are included in following table:
<table>
<thead>
<tr>
<th>Table 2 Results of $NP\tilde{W}$</th>
<th>Fuzzy Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_0$ as a Fuzzy Number</td>
<td>$\tilde{F}_0$</td>
</tr>
<tr>
<td>EUAB as a Fuzzy Number</td>
<td>$EUAB$</td>
</tr>
<tr>
<td>MARR as a Fuzzy Number</td>
<td>$\tilde{\iota}$ (%)</td>
</tr>
<tr>
<td>Salvage Value as a Fuzzy Number</td>
<td>$\tilde{S}$</td>
</tr>
<tr>
<td>Useful Life as a Fuzzy Number</td>
<td>$\tilde{n}$</td>
</tr>
<tr>
<td>Each Cost Data as a Fuzzy Number</td>
<td>$\tilde{F}_0$</td>
</tr>
<tr>
<td>$EUAB$</td>
<td>180</td>
</tr>
<tr>
<td>$\tilde{\iota}$</td>
<td>9</td>
</tr>
<tr>
<td>$\tilde{S}$</td>
<td>90</td>
</tr>
<tr>
<td>$\tilde{n}$</td>
<td>18</td>
</tr>
<tr>
<td>$NP\tilde{W}$</td>
<td>300.05</td>
</tr>
<tr>
<td>Difference (%)</td>
<td>-58.2</td>
</tr>
<tr>
<td>Single-valued estimates NPW</td>
<td>$F_0$</td>
</tr>
<tr>
<td>1000</td>
<td>200</td>
</tr>
</tbody>
</table>

The results can also be plotted as a line graph in Figure 2.
Examine the Table 2 and the Fig. 2, the following observations can be made:

- EUAB and MARR are the most sensitive to the changes in the cost data, followed by first cost, useful life, and salvage value.
- Because division and power factor are involved in the fuzzy number operation rules, the percentage differences are changing faster in increasing direction than in decreasing direction for the MARR.
- The percentage difference is changing linearly for first cost, EUAB, and salvage value.
- The total difference is not equal to the sum of each individual difference for each cost data.

4.3 Effects of Alpha - Cut

In order to reduce the range expansion of fuzzy set, decision makers can apply the $\alpha$-cut to reduce the final fuzzy set. The approach of $\alpha$-cut is to decrease the range of the final fuzzy set by increasing the degree of membership. In normal fuzzy sets, the degree of membership is from 0 to 1. If it is desirable to reduce the range of the final fuzzy set, decision makers can increase the degree of membership from 0 to $\alpha$, $\alpha \in [0,1]$. In this section, the range reduction will be discussed by different $\alpha$ values.
According to the Fig. 3, the values of \( a' \) and \( d' \) are the NPW associated to the value of \( \alpha \). Because the value \( a \) to \( b \) and \( c \) to \( d \) is linear increasing and decreasing respectively, the value of \( a' \) and \( d' \) can be obtained by interception from \( a \) to \( b \) and \( c \) to \( d \). The values of \( b \) and \( c \) are the same because their degrees of memberships are always equal to 1. To illustrate the effect of \( \alpha \)-cut, the range reduction for the fuzzy numbers \( a \) and \( d \) is calculated by assigning \( \alpha \) from 0.1 to 0.9 to each fuzzy cost data in Table 1. Based on the effects of \( \alpha \)-cut, decision makers can assign a different \( \alpha \) value according to different final fuzzy set. If decision makers have more confidence with the fuzzy set of an alternative, a higher value of \( \alpha \) can be assigned to the final fuzzy set.

In general, the effect of \( \alpha \)-cut can be expressed as following formulas:

\[
NP\tilde{W}_{a,\alpha} = NP\tilde{W}_a + \alpha(NP\tilde{W}_b - NP\tilde{W}_a)
\]

\[
NP\tilde{W}_{d,\alpha} = NP\tilde{W}_d - \alpha(NP\tilde{W}_d - NP\tilde{W}_c)
\]

By the above formulas, the effects of \( \alpha \)-cut can be illustrated as in Fig. 4.

![Fig. 4 Alpha v.s. Percentage Reduction](image)

From the Fig. 4, it is obvious that the percentage reduction of fuzzy range for each cost data by \( \alpha \)-cut is not a linear representation. When each cost data is a fuzzy number, \( \alpha \)-cut can have a significant effect to reduce the final fuzzy range.

There are two ways to reduce the expansion of final fuzzy range, one is sensitivity analysis and the other is \( \alpha \)-cut. After the sensitivity analysis, decision makers can have the information about the sensitivity for each cost data. In order to minimize the disturbance caused by the fuzzy range increasing, decision makers should focus on the most sensitive cost data.
Regardless the attempts to reduce the range on cost data, the final fuzzy set range may still be very large. For this situation, decision makers can apply the $\alpha$-cut to reduce the range. Depending on the different cases, decision makers can choose different value of $\alpha$ (from 0 to 1) to reduce the range. If decision makers are very confident with each cost data, a higher value of $\alpha$ can be assigned.

5. Conclusion

Considering the trapezoidal fuzzy method, it is a good approach to deal with cash flow analysis. From the data analysis, it can be concluded that the trapezoidal fuzzy method is suitable in Engineering Economic. With this approach, decision makers can have more space to define each cost data and can obtain more information from the final result. Though there are still some inconvenient in applying this new approach, such as expansion of fuzzy set after operation and requirement of a ranking method, those situations can be solved if decision makers are careful to define the fuzzy sets for each cost data and select a suitable ranking method.

REFERENCES


