Pricing European and American options of real estate index

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Abstract
In this paper, we modeled the real estate index options by no-arbitrage approach and obtained the partial
differential equations (PDEs) of the real estate index options. Then we proposed a novel radial basis
function (RBF) to solve the PDEs.

Keywords: Real Estate Index Options, No-arbitrage Approach, Radial Basis Functions

INTRODUCTION

Real estate derivatives market has emerged more than 20 years. However, in the beginning,
little attention was paid to this market. Although the real estate derivatives market has displayed a
rapid development for many years, the trading volume and liquidity are not comparable to financial
derivatives because of investors’ low acceptance of the real estate derivatives as new hedging
instruments which can be partly attributed to the lack of reliable pricing models. In recent years,
the pricing of real estate derivatives has attracted more and more attention. Buttimer et al. (1997)
employed a bivariate binominal model to price the total return swap contingent on a real estate
index and interest rate. They found a positive but negligible swap spread price. Bjork and Clapham
(2002) revised the model proposed by Buttimer et al. (1997) and proved that the theoretical price
of the total return swap is equal to zero when using the no-arbitrage approach. However, all the
no-arbitrage models above ignore the fact that the real estate index is non-tradable. To solve this
pricing problem, Geltner and Fisher (2007) proposed an equilibrium model for the pricing of real
estate forwards and total return swap contracts. Fabozzi et al. (2012) considered the econometric
properties of real estate indices and the incompleteness of real estate market by using the real estate futures market to complete such market. Under the consumption that the market price of the real estate index risk is known, the closed-form solutions were obtained for futures, European options and total return swaps contingent on real estate index. In this paper, we modeled the real estate index options by no-arbitrage approach and obtained the PDEs of the real estate index options. Then we develop a novel approach for solving the PDEs.

The common numerical approaches for options pricing are the binomial and trinomial trees, the finite difference (Hull and White 1990, O'Sullivan and O'Sullivan 2011), the finite element and finite volume methods (Zvan et al. 2001, Tangman et al. 2008). In addition, many researchers also used the meshfree methods which are effective to solve PDEs. Point interpolation method (PIM) is the most commonly used meshfree method and has been achieved remarkable progress in recent years. Polynomials basis function (PBF) is one of the earliest interpolation schemes. But, the method has a problem that polynomial basis possibly cause singularity (Liu and Gu 2005). After radial basis functions (RBFs) proposed, the shortcoming of PBFs has been improved largely. In this paper, we combined thin plate splines (TPSs) and PBFs to construct the RBFs.

REAL ESTATE INDEX OPTION PROBLEM

Real estate indices exhibit a positive autocorrelation in the short term and a negative autocorrelation in the long term. Therefore, we employed a mean reverting stochastic model to measure the real estate index movement (Fabozzi et al. 2012).

\[ dY_t = \left[ \frac{d\psi_t}{dt} - \theta(Y_t - \psi_t) \right] dt + \sigma dW_t \] (1)

Where \( Y_t = \log(X_t) \), the underlying asset \( X_t \) is the real estate index, \( \psi_t \) is the long run mean trend of real estate indices in log scale, and \( \theta \) is the mean-reversion speed parameter.

Let \( V(Y_t, t) \) denote the option price, we have

\[ dV(Y_t, t) = \left( \frac{\partial V(Y_t, t)}{\partial t} + \mu \frac{\partial V(Y_t, t)}{\partial Y_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V(Y_t, t)}{\partial Y_t^2} \right) dt + \sigma \frac{\partial V(Y_t, t)}{\partial Y_t} dW_t \] (2)

Hence

\[ \frac{dV}{V} = \mu_v(Y_t, t) dt + \sigma_v(Y_t, t) dW_t \] (3)
Considering that we construct a riskless hedging portfolio $\pi$ that contains two derivatives of different maturities $T_1$ and $T_2$. Let $\Delta$ denote the number of units the derivatives. The portfolio is

$$\pi = V_1 - \Delta V_2$$

(4)

In a very short period of time $dt$, the value changes of the portfolio is

$$d\pi = [V_1 \mu_{Y_i}(Y_i,t) - \Delta V_2 \mu_{Y_i}(Y_i,t)] dt$$

(5)

Under no arbitrage conditions, we have

$$\mu_{Y_i}(Y_i,t) - r = \lambda(Y_i,t) \sigma_{Y_i}(Y_i,t)$$

(6)

where $\lambda(Y_i,t)$ is the market price of risk.

Finally, we obtained the following PDE.

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 X_i^2 \frac{\partial^2 V}{\partial X_i^2} + \left( \mu - \lambda \sigma + \frac{1}{2} \sigma^2 \right) X_i \frac{\partial V}{\partial X_i} - rV = 0$$

(7)

We define following real estate index options Operator, so that we can easily describe in the section of American options.

$$LV = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 X_i^2 \frac{\partial^2 V}{\partial X_i^2} + \left( \mu - \lambda \sigma + \frac{1}{2} \sigma^2 \right) X_i \frac{\partial V}{\partial X_i} - rV$$

(8)

**European Options**

For the sake of simplicity, we restrict our attention to put options, because call option can be treated in perfect analogy. Consider a put option with maturity $T$ and strike price $E$. The final condition is

$$V_p(X_T, T) = g_p(X_T, T) = (K - X)^+$$

(9)
Boundary conditions are

\[ V_p(0, t) = K \exp(-r(T-t)) \]
\[ \lim_{X \to \infty} V_p(X, t) = 0 \]  

(10)

**American Options**

We still consider a put option, the final condition is

\[ V_p(X_T, T) = g_p(X_T, T) = (K-X)^+ \]  

(11)

The American option price \( V \) is not less than \( g_p \). Hence, the equation is

\[ \begin{cases} LV \geq 0, & V \geq g_p \\ (LV)(V-g_p) = 0 \end{cases} \]  

(12)

Boundary conditions are

\[ V_p(0, t) = K \]
\[ \lim_{X \to \infty} V_p(X, t) = 0 \]  

(13)

**TPS-PBF APPROXIMATION**

The TPSs do not involve any free shape parameter. In particular, we employ the very popular TPSs of second order, which are

\[ R_i(x) = (x-x_i)^4 \log(|x-x_i|) \quad i = 1, 2, \ldots, n \]  

(14)

Combined with PBFs is

\[ u(x) = R^T(x)a + P^T(x)b \]  

(15)
Where \( a \) and \( b \) are the coefficient vector of radial basis \( R \) and polynomial basis \( P \) respectively. The polynomial term has to satisfy an extra requirement that guarantees unique approximation of a function, which is the following constraint.

\[
P^T a = 0
\]

We have the following system of linear equations

\[
\begin{bmatrix}
R & P \\
P^T & 0
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix} =
\begin{bmatrix}
a \\
b
\end{bmatrix}
= 
\begin{bmatrix}
U \\
0
\end{bmatrix}
\]

(16)

The matrix \( R \) is non-singular, therefore

\[
\begin{bmatrix}
a \\
b
\end{bmatrix} = G^{-1}
\begin{bmatrix}
U \\
0
\end{bmatrix}
\]

(17)

Equation (15) can be re-written as

\[
u(x) = \left[ R^T(x) \quad P^T(x) \right] G^{-1}
\begin{bmatrix}
U \\
0
\end{bmatrix} = \Phi(x) \overline{U}
\]

(18)

**TPS-PBF for Options**

Employing TPS-PBFs to approximate the real estate index option price \( V \), we have

\[
V(X,t) = \Phi(X) \overline{U}
\]

(19)

In order to numerically handle the unboundedness of the \( X \)-domain, we use the following change to variables

\[
S = 1 - \exp(-X/L)
\]

(20)
Where $L$ is a parameter, which can be determined by equation $1 - \exp(-E/L) = 0.7$.

Then equation (7) can be rewritten as

$$
\frac{\partial V}{\partial t} + A(S)\frac{\partial^2 V}{\partial S^2} + B(S)\frac{\partial V}{\partial S} - rV = 0
$$

(21)

The RBF proposed in this paper is independent with time and determined the underlying asset $S$. Therefore, the equation (21) is continuously differentiable on $S$, and the following equation can be obtained.

$$
\Phi V' = (-A(S)\Phi_{SS} - B(S)\Phi_S + r\Phi)V = DV
$$

(22)

**European Options**

In this paper, we divided the time interval $[0, T]$ into $M$ points, therefore $V^k(S) = V(S, k\Delta t), k = 1, 2, \cdots, M$. Combined with forward and backward difference scheme discretize the time derivative, we obtain the weight implicit scheme.

$$
\Phi V^{k+1}_1 = \Phi V^k + \Delta tD\left[\theta V^{k+1} + (1 - \theta) V^k\right]
$$

(23)

Let $P_1 = [\Phi - \theta\Delta tD]$ and $P_2 = [\Phi + (1 - \theta)\Delta tD]$, equation (23) can rewrite as:

$$
P_1 V^{k+1} = P_2 V^k
$$

(24)

Due to the non-smoothness of the options’ payoff, the Crank-Nicolson scheme fails to achieve its usual second-order accuracy. Therefore, we use the implicit Euler scheme, which is unconditionally stable and allows us to smooth the discontinuities of the options’ payoffs.

**American Options**
Pricing American option is complicated because we must face a free boundary problem. Bermudan options allow early exercises at a finite number of pre-specified exercise times which is similar to American options. By increasing the number of exercise times we see that the Bermudan put option pricing is closely related to the American put option pricing. So we approximate the price of the American option with the price of a Bermudan option (Khaliq, Voss et al. 2006, Lim, Lee et al. 2014). According to equations (11)(12)(13), we consider an option which can be exercised not on the whole time interval \([0, T]\), only at the dates \(t_1, t_2, \ldots, t_M\).

Now, we assume that in each time interval \((t_k, t_{k+1}), k = 1, 2, \ldots, M\) the relations (11)(12)(13) holds true. That is we consider the problems

\[
\begin{align*}
LV &= 0 \\
V_p(0, t) &= K, \quad \lim_{X \to \infty} V_p(X, t) = 0
\end{align*}
\]

(25)

which hold true in the time intervals \((t_k, t_{k+1}), k = 1, 2, \ldots, M\). So the third relations of (12) is automatically satisfied in every time interval. The second relation of (12) is constraint only at times \(t_1, t_2, \ldots, t_M\)

\[
V(X, t_k) = \max\left(\lim_{t \to \infty} V(t, X), g_p(X)\right), \quad k = 1, 2, \ldots, M
\]

(26)

Therefore, the solution of the American options is that

\[
\begin{align*}
P^k V^{k+1} &= P^k V^k \\
V^k &= \max(V^k, V^M)
\end{align*}
\]

(27)

where \(V^M\) is the option prices of maturity.

**NUMERICAL RESULTS**

Let \(V_{RPBP}(X_i, 0)\) denote the prices of using point interpolation method, \(V(X_i, 0)\) imply the real prices of relative options. We defined the following error formulas
\[ \text{MaxError} = \max_{i=1,2,...,N} \left| V_{\text{RBPF}}(X_i,0) - V(X_i,0) \right| \] (28)

The mean square norm is

\[ \text{RmsError} = \frac{1}{n} \sqrt{\sum_{i=1}^{n} \left( V_{\text{RBPF}}(X_i,0) - V(X_i,0) \right)^2} \] (29)

**European Options**

Table 1 give the numerical results of put options when the strike price is 1800 and time to maturity is 1 year, risk-free rate is 0.04. The left part of Table 2 is the result by only TPS basis functions, and the right is obtained by the scheme of TPS-PBF. N is the number of scatter nodes in the domain, and Cond is the condition numbers of System matrix. As we can see, the option price can be computed with an error of order 10^-2 in the maximum norm and 10^-2 in the mean square norm. So the levels of accuracy is very high in this paper. Some researchers may suspect that the accuracy is not precise compared with stock options, which is understandable because they forgot the strike price is 1800 rather than 10 (Wilmott et al. 1995, Hon 2002).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \text{RmsError} )</th>
<th>( \text{MaxError} )</th>
<th>Cond/G</th>
<th>( \text{RmsError} )</th>
<th>( \text{MaxError} )</th>
<th>Cond/G</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1.7741E-02</td>
<td>5.4920E-01</td>
<td>3.0547E+08</td>
<td>1.1902E-02</td>
<td>2.6035E-01</td>
<td>7.1575E+09</td>
</tr>
<tr>
<td>75</td>
<td>6.2935E-03</td>
<td>2.8950E-01</td>
<td>2.5354E+09</td>
<td>4.6015E-03</td>
<td>1.7662E-01</td>
<td>6.0535E+10</td>
</tr>
<tr>
<td>100</td>
<td>3.4699E-03</td>
<td>1.9527E-01</td>
<td>1.1128E+10</td>
<td>2.8463E-03</td>
<td>1.3726E-01</td>
<td>2.6888E+11</td>
</tr>
<tr>
<td>125</td>
<td>2.4002E-03</td>
<td>1.4825E-01</td>
<td>3.4825E+10</td>
<td>2.1322E-03</td>
<td>1.1361E-01</td>
<td>8.4659E+11</td>
</tr>
<tr>
<td>150</td>
<td>1.9059E-03</td>
<td>1.2034E-01</td>
<td>8.8030E+10</td>
<td>1.7784E-03</td>
<td>9.7544E-02</td>
<td>2.1505E+12</td>
</tr>
<tr>
<td>200</td>
<td>1.4458E-03</td>
<td>8.8427E-02</td>
<td>3.7845E+11</td>
<td>1.4084E-03</td>
<td>7.6763E-02</td>
<td>9.2978E+12</td>
</tr>
<tr>
<td>250</td>
<td>1.2321E-03</td>
<td>7.1015E-02</td>
<td>1.1687E+12</td>
<td>1.2223E-03</td>
<td>6.3786E-02</td>
<td>2.8812E+13</td>
</tr>
<tr>
<td>300</td>
<td>1.1288E-03</td>
<td>5.8820E-02</td>
<td>2.9312E+12</td>
<td>1.0919E-03</td>
<td>5.4450E-02</td>
<td>7.2426E+13</td>
</tr>
</tbody>
</table>

It’s obviously that, with the increasing of scatter nodes, the results become more and more precise. But with the increasing of scatter nodes the condition numbers of system matrix increase fast, this could cause the poor precision. Compared with only TPS basis functions, the novel method can effectively improve the results accuracy.

**American Options**
Let us considering American option which strike price is 1500 and time to maturity is 1 year, the results shown in table 2. We can see the same properties with European options, that with the number of scatter nodes increase the price tends to a stable value.

<table>
<thead>
<tr>
<th>N</th>
<th>1450</th>
<th>1475</th>
<th>1500</th>
<th>1525</th>
<th>1550</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>75.9123</td>
<td>63.3945</td>
<td>52.0709</td>
<td>43.4651</td>
<td>35.5482</td>
</tr>
<tr>
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<td>75.5831</td>
<td>63.2658</td>
<td>52.5959</td>
<td>43.3723</td>
<td>35.4823</td>
</tr>
<tr>
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<td>62.9376</td>
<td>52.1276</td>
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<td>35.4978</td>
</tr>
<tr>
<td>175</td>
<td>75.7149</td>
<td>63.1323</td>
<td>52.3914</td>
<td>43.2718</td>
<td>35.5705</td>
</tr>
<tr>
<td>200</td>
<td>75.6216</td>
<td>62.9817</td>
<td>52.1525</td>
<td>43.1305</td>
<td>35.4760</td>
</tr>
<tr>
<td>225</td>
<td>75.6087</td>
<td>62.9762</td>
<td>52.3057</td>
<td>43.1494</td>
<td>35.4586</td>
</tr>
<tr>
<td>250</td>
<td>75.4967</td>
<td>62.9867</td>
<td>52.1631</td>
<td>43.1441</td>
<td>35.3946</td>
</tr>
<tr>
<td>275</td>
<td>75.5347</td>
<td>62.9614</td>
<td>52.2637</td>
<td>43.1216</td>
<td>35.4541</td>
</tr>
<tr>
<td>300</td>
<td>75.5386</td>
<td>62.9358</td>
<td>52.1700</td>
<td>43.1063</td>
<td>35.4352</td>
</tr>
</tbody>
</table>

**CONCLUSIONS**

In developed countries, the real estate property have reached 30% to 40% of total market assets. However, the risk management tools available for hedging real-estate risk are very much in their infancy and have problems ranging from illiquidity of trading to lack of theoretical development in terms of modelling. In this paper, we advocated a suitable framework for pricing real estate options and proposed a novel method to solve the PDEs of European and American options.

The RBPI approach developed in this paper offers several advantages over the more conventional RBF approximation. First, the scheme combined TPSs and PBFs, which improves the accuracy and efficiency of the result compared with the pure RBFs. Second, we replace the original domain with a finite one, and don’t introduce unknown finite boundaries and prescribe artificial conditions as in the previous methods. That is our innovation of variables change. Finally, we employed a local mesh refinement strategy, which allows us to easily and effectively handle the non-smoothness of the options’ payoff. The results shown that the option prices can be computed with an error of order 10e-2 in the maximum norm and 10e-2 in the mean square norm, which is very precise respect to the options of large strike price.

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Bibliography


