# State and Parameter Estimation for a Partially Observable System Subject to Random Failure

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**Abstract:** We consider a partially observable system subject to random failure. The system is monitored at equidistant points of time and the information obtained is stochastically related to the system's state which is unobservable, except the failure state. By combining the failure information and the information obtained through condition monitoring and applying the change of measure method, we derive recursive filters and develop a procedure for parameter estimation based on the EM algorithm. The procedure is tested using a real data set.

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## *Introduction*

We consider a partially observable system subject to random failure. The state of the system is described by a homogeneous Markov chain with a finite state space. The system's failure is observable, but the working state is not. The information about the system, described by an observation process, is obtained at equally spaced inspection times. Condition monitoring techniques have been well developed and are frequently used in practice. The most commonly used methods include spectrometric analysis of engine oil and vibration monitoring. However, there are still very few mathematical models capable of utilizing the information obtained on-line for an effective maintenance decision-making.

Several types of maintenance models with partial information have appeared in the literature focusing mainly on the decision aspect, e.g., see Sondik (1971), Jensen and Hsu (1993), Stadje (1994), Aven (1996), and Makis, *et*. *al*. (1998). The condition-based maintenance (CBM) models applicable to real situations include the state-space model for furnace erosion prediction by Christer, *et*. *al*. (1997) and the proportional hazards model by Makis and Jardine (1992).

The papers by Makis and Jardine (1991, 1992) provided a theoretical basis for the development of a software called EXAKT (EXAKT Manual, 2000) by the CBM laboratory established at the University of Toronto in 1994 and supported by several major industrial organizations, MMO and NSERC. The EXAKT software is now well developed and the CBM laboratory has had rich experience in analyzing CBM data and providing consultancy to the maintenance engineers using the software.

In this paper, we study the estimation problem in a partially observable system described by a hidden Markov model (HMM) with observable failure state. HMM has attracted considerable attention in the past two decades. However, there has been little attempt to apply HMM to maintenance/replacement problems. To the authors' knowledge, the work by Fernández-Gaucherand, *et*. *al*. (1993) is the only published paper that studied both estimation and control in a simple replacement model described by a two state HMM with unobservable failure state. With the rapid development of fast computers, well developed information systems in many companies and the implementation of sophisticated condition-monitoring techniques, HMM will likely attract considerable attention in the maintenance/replacement area in the near future.

There are two main estimation problems in an HMM: state and parameter estimation. The estimate based on processing past and present observation is called filter. In this paper, we apply the change of measure method (Elliott, *et. al.*, 1995) to obtain the state and parameter estimates of the (partially) hidden Markov process describing the state of the system. The paper is organized as follows. We formulate the model of the deteriorating system in next section. Then, a new measure is constructed, under which the observations are i.i.d. Under the new measure, we derive a general filter and obtain as special cases filters for state, transitions and occupation times. Based on these filters, parameter re-estimation using the EM algorithm is discussed. Finally, a numerical example illustrating the procedure is given.

## *The Model*

We consider a deteriorating system subject to random failure. The system is (partially) observable at times $k\Delta,\ k = 1, 2, \cdots$, and the observation process $(Y)$ is stochastically related to the state process $(X)$. Suppose all stochastic processes are defined on a probability space $(\Omega, \mathcal{F}, P)$. Denote the state of the system at time $k\Delta$ by $X_k,\ k = 0,1,2,\cdots$. The state of the system at time 0, $X_0$, is given, or its distribution is known. The state process $\{X_k,\ k = 0,1,2,\cdots\}$ is described by a homogeneous Markov chain with a finite state space $S_X = \{e_1, e_2, \cdots, e_N\}$, where $e_i$ is a unit $N$-dimensional vector whose $i$th element is 1. $e_i$ represents the $i$th possible state of the system. The last state $e_N$ represents the failure state of the system and this state is observable. The first $N-1$ states $\{e_1, \cdots, e_{N-1}\}$ are not observable. We assume that the failure state $e_N$ is an absorbing state, i.e., the system cannot be recovered to any working state when it fails. Let $\tilde{A} = (a_{ji})_{N\times N}$ be the transition probability matrix of the Markov chain $X_k,\ k = 0,1,2,\cdots$ and $A$ be the matrix consisting of the first $N-1$ columns of $\tilde{A}$, where

$$a_{ji} = P\{X_{k+1} = e_j \mid X_k = e_i\}, \quad i, j = 1, 2, \cdots, N. \tag{1}$$

Obviously, $a_{jN} = 0$ for $j = 1, 2, \cdots, N-1$ and $a_{NN} = 1$, i.e., $\tilde{A} = (A, e_N)$.

Let $Y_k$ be the measurement taken at time $k\Delta$ given that the system is operating. When the system fails, only the failure information is available. Suppose $Y_k$ has $M$ possible values, $Y_k \in S_Y = \{f_1, f_2, \cdots, f_M\}$ where $f_i$ is a unit $M$-dimensional vector with 1 in the $i$th position. Suppose further that $P\{Y_k = f_j \mid X_0, X_1, \cdots, X_k, Y_1, \cdots, Y_{k-1}\} = P\{Y_k = f_j \mid X_k\}$. Write $C = (c_{ji})_{M\times(N-1)}$ where

$$c_{ji} = P\{Y_k = f_j \mid X_k = e_i\}, \quad i = 1, 2, \cdots, N-1,\ j = 1, 2, \cdots, M. \tag{2}$$

Based on the above assumptions, the information available consists of two parts: the measurement information $(Y)$, and the failure information. Write $g_i = (f_i' \quad 0)'$, $i = 1, 2 \cdots, M$, and let $g_{M+1}$ be an $(M+1)$-dimensional vector with 1 in the last position and 0 elsewhere. To combine the observed measurement information and the failure information, we define a new random variable $Z_k \in S_Z = \{g_1, g_2, \cdots, g_{M+1}\}$ such that

$$P\{Z_k = g_j \mid X_k = e_i\} = P\{Y_k = f_j \mid X_k = e_i\} = c_{ji}, \quad 1 \le i < N, 1 \le j \le M,$$

$$P\{Z_k = g_j \mid X_k = e_i\} = \begin{cases} 0, & 1 \le i < N,\ j = M+1 \text{ or } i = N, 1 \le j \le M \\ 1, & i = N,\ j = M+1 \end{cases}.$$

Thus, $Z_k$ represents the overall information available at time $k\Delta$. When $Z_k = g_j$ $(1 \le j \le M)$, the system is in some working state and the measurement is $Y_k = f_j$. When $Z_k = g_{M+1}$, the system is in the failure state and no measurement is available.

Let $\mathcal{F}_k$, $\mathcal{Z}_k$ and $\mathcal{G}_k$ be the complete filtrations generated by $(X_0, X_1, \cdots, X_k)$, $(Z_1, Z_2, \cdots, Z_k)$ and $(X_0, X_1, \cdots, X_k, Z_1, Z_2, \cdots, Z_k)$, respectively. Then, the system dynamics is described by the following HMM:

$$X_{k+1} = \tilde{A} X_k + V_{k+1}, \quad k = 0, 1, 2, \cdots, \tag{3}$$

$$Z_k = \tilde{C} X_k + W_k, \quad k = 1, 2, \cdots, \tag{4}$$

where $X_k \in S_X$, $Z_k \in S_Z$, $\tilde{A} = (a_{ji})_{N \times N}$, $\tilde{C} = (c_{ji})_{(M+1) \times N}$ and their elements satisfy

$$\sum_{j=1}^{N} a_{ji} = 1, \quad a_{ji} \ge 0, \quad a_{jN} = \begin{cases} 0, & 1 \le j \le N-1 \\ 1, & j = N \end{cases}, \tag{5}$$

$$\sum_{j=1}^{M} c_{ji} = 1, \quad c_{ji} \ge 0, \quad i = 1, 2, \cdots, N-1, \tag{6}$$

$$c_{ji} = \begin{cases} 0, & 1 \le i < N,\ j = M+1 \text{ or } \mathrm{i = N, 1 \le j \le M} \\ 1, & i = N,\ j = M+1 \end{cases}. \tag{7}$$

$V_{k+1}$ and $W_k$ are mutually independent martingale increments satisfying $E[V_{k+1} \mid \mathcal{F}_k] = 0$, $E[W_k \mid \mathcal{F}_k, \mathcal{G}_{k-1}] = 0$,

$$\langle V_{k+1} \rangle = E[V_{k+1} V_{k+1}' \mid X_k] = \mathrm{diag}(\tilde{A} X_k) - \tilde{A} \mathrm{diag}(X_k) \tilde{A}', \tag{8}$$

$$\langle W_k \rangle = E[W_k W_k' \mid X_k] = \mathrm{diag}(\tilde{C} X_k) - \tilde{C} \mathrm{diag}(X_k) \tilde{C}', \tag{9}$$

where $\mathrm{diag}(z)$ is the diagonal matrix with vector $z$ on its diagonal.

## *Change of Measure*

We begin with defining a new probability measure $\overline{P}$ such that under this measure, the observation sequence is i.i.d. and $(X)$ is a Markov chain with the transition matrix $\tilde{A}$. Then, we construct measure $P$, under which the system's dynamics is described by (3) and (4). It turns out to be much easier to derive the filters under the new measure and then transform the results back to the real situation, under measure *P*. We use prime to denote transpose and $\langle a,b\rangle = a'b$ to denote inner product. Further, we introduce the following notation: $X_k^i = \langle X_k, e_i\rangle$, $Z_k^i = \langle Z_k, g_i\rangle$, $c_k^i = \langle \tilde{C}X_k, g_i\rangle$, $\overline{\lambda}_k = (M+1)\sum_{i=1}^{M+1} c_k^i Z_k^i$ and $\overline{\Lambda}_k = \prod_{l=1}^{k} \overline{\lambda}_l$, for $k = 1,2,\cdots$. From Kolmogorov's extension theorem, there exists a new probability measure *P* such that $\left.\frac{dP}{d\overline{P}}\right|_{\mathcal{G}_k} = \overline{\Lambda}_k$. The results showing that the state process and the observation process follow (3) and (4) are summarized in the following lemmas.

**Lemma 1** *Under P*, $E[X_{k+1} \mid \mathcal{G}_k] = \tilde{A}X_k$.

**Lemma 2** *Under P*, $E[Z_{k+1} \mid X_{k+1}] = \tilde{C}X_{k+1}$.

## *Filters*

For any $\mathcal{G}$-adapted scalar process $\phi_k$, define $\gamma_k(\phi_k) = \overline{E}[\overline{\Lambda}_k \phi_k X_k \mid \mathcal{Z}_k]$. The estimate of $\phi_k$, denoted by $\hat{\phi}_k$, is given by

$$\hat{\phi}_k = E[\phi_k \mid \mathcal{Z}_k] = \frac{\overline{E}[\overline{\Lambda}_k \phi_k \mid \mathcal{Z}_k]}{\overline{E}[\overline{\Lambda}_k \mid \mathcal{Z}_k]}.$$

We call $\overline{E}[\overline{\Lambda}_k \phi_k \mid \mathcal{Z}_k]$ the unnormalized estimate of $\phi_k$, and $\overline{E}[\overline{\Lambda}_k \mid \mathcal{Z}_k]$ the normalizing factor. Clearly $\langle \gamma_k(\phi_k), \mathbf{1}\rangle = \overline{E}[\overline{\Lambda}_k \phi_k \langle X_k, \mathbf{1}\rangle \mid \mathcal{Z}_k] = \overline{E}[\overline{\Lambda}_k \phi_k \mid \mathcal{Z}_k]$, where **1** is a vector with all elements 1. Therefore, we only need to obtain $\gamma_k(\phi_k)$ and $\gamma_k(1)$ to calculate $\hat{\phi}_k$.

Suppose $H_k$ is a scalar process of the form

$$H_{k+1} = H_k + \alpha_{k+1} + \langle \beta_{k+1}, X_{k+1}\rangle + \langle \delta(Z_{k+1}), X_{k+1}\rangle,$$

where $\alpha_{k+1}$, $\beta_{k+1}$, are $\mathcal{G}_k$ measurable, $\delta(Z_{k+1})$ is a known vector function of $Z_{k+1}$, $\alpha_{k+1}$ is a scalar, $\beta_{k+1}$ and $\delta(Z_{k+1})$ are vectors of dimension *N*. Write $d_j(Z_{k+1}) = \sum_{i=1}^{M+1} Mc_{ij} Z_{k+1}^i e_j$.

**Theorem 1** *The recursive formula for the filter of* $H_k$ *is*

$$\gamma_{k+1}(H_{k+1}) = \sum_{j=1}^{N} d_j(Z_{k+1}) \sum_{l=1}^{N} a_{jl} \{\langle \gamma_k(H_k), e_l\rangle + \overline{E}[(\alpha_{k+1} + \langle \beta_{k+1}, e_j\rangle + \langle \delta(Z_{k+1}), e_j\rangle)\overline{\Lambda}_k X_k^l \mid \mathcal{Z}_{k+1}]\}.$$

By applying Theorem 1 with some specific $\alpha_{k+1}$, $\beta_{k+1}$ and $\delta(Z_{k+1})$, we obtain as special cases filters for states, transitions and occupation times.

### Filter for the state

Let $q_k(e_r) = \overline{E}[\overline{\Lambda}_k \langle X_k, e_r\rangle \mid \mathcal{Z}_k]$, $\quad q_k = (q_k(e_1), q_k(e_2), \cdots, q_k(e_N))' = \gamma_k(1)$. Then the recursive formula for calculating $q_k$, the unnormalized state filter, is

$$q_{k+1} = \sum_{j=1}^{N} d_j(Z_{k+1}) \sum_{l=1}^{N} a_{jl} \langle q_k, e_l \rangle . \quad (10)$$

**Filter for the number of jumps**

Let $\mathcal{J}_k^{rs} = \sum_{l=1}^{k} \langle X_{l-1}, e_r \rangle \langle X_l, e_s \rangle$ be the number of jumps from state $e_r$ to state $e_s$ up to time *k*, for $r = 1,2,\cdots,N-1$ and $s = 1,2,\cdots,N$. The recursive formula for the filter of $\mathcal{J}_k^{rs}$ is

$$\gamma_{k+1}(\mathcal{J}_{k+1}^{rs}) = \sum_{j=1}^{N} d_j(Z_{k+1}) \sum_{l=1}^{N} a_{jl} \langle \gamma_k(\mathcal{J}_k^{rs}), e_l \rangle + d_s(Z_{k+1}) a_{sr} . \quad (11)$$

**Filter for the occupation time**

Let $\mathcal{O}_k^{r} = \sum_{l=1}^{k} \langle X_{l-1}, e_r \rangle$ be the number of occasions up to time $k-1$ that the system has been in working state $e_r$, for $r = 1,2,\cdots,N-1$. The recursive formula for the filter of $\mathcal{O}_k^{r}$ is

$$\gamma_{k+1}(\mathcal{O}_{k+1}^{r}) = \sum_{j=1}^{N} d_j(Z_{k+1}) \sum_{l=1}^{N} a_{jl} \langle \gamma_k(\mathcal{O}_k^{r}), e_l \rangle + \sum_{j=1}^{N} d_j(Z_{k+1}) a_{jr} \langle q_k, e_r \rangle . \quad (12)$$

For the application of the EM algorithm in the next section, we would need another version of the occupation time defined by $\tilde{\mathcal{O}}_k^{r} = \sum_{l=1}^{k} \langle X_l, e_r \rangle$, which counts the number of occasions from time 1 up to time *k* that the system has been in the working state $e_r$, for $r = 1,2,\cdots,N-1$. The recursive formula for the filter of $\tilde{\mathcal{O}}_k^{r}$ is

$$\gamma_{k+1}(\tilde{\mathcal{O}}_{k+1}^{r}) = \sum_{j=1}^{N} d_j(Z_{k+1}) \sum_{l=1}^{N} a_{jl} \langle \gamma_k(\tilde{\mathcal{O}}_k^{r}), e_l \rangle + d_r(Z_{k+1}) \sum_{l=1}^{N} a_{rl} \langle q_k, e_l \rangle . \quad (13)$$

**Filter for the state to observation transitions**

Let $\mathcal{T}_k^{rs} = \sum_{l=1}^{k} \langle X_l, e_r \rangle \langle Z_l, g_s \rangle$ be the number of times up to time *k* that the observation process is in state $f_s$ given the system is in the working state $e_r$, for $r = 1,2,\cdots,N-1$ and $s = 1,2,\cdots,M$. The recursive formula for the filter of $\mathcal{T}_k^{rs}$ is

$$\gamma_{k+1}(\mathcal{T}_{k+1}^{rs}) = \sum_{j=1}^{N} d_j(Z_{k+1}) \sum_{l=1}^{N} a_{jl} \langle \gamma_k(\mathcal{T}_k^{rs}), e_l \rangle + Z_{k+1}^{s} c_{sr} e_r \sum_{l=1}^{N} a_{rl} \langle q_k, e_l \rangle . \quad (14)$$

## *Parameter re-estimation*

The hidden Markov model described in this paper is determined by the set of parameters

$$\theta = \{a_{ji},\ i = 1,2,\cdots,N-1,\ j = 1,2,\cdots,N; \quad c_{ji},\ i = 1,2,\cdots,N-1,\ j = 1,2,\cdots,M\}$$

satisfying constraints

$$\sum_{j=1}^{N} a_{ji} = 1, \quad a_{ji} \geq 0, \quad i = 1,2,\cdots,N-1,\ j = 1,2,\cdots,N, \tag{15}$$

$$\sum_{j=1}^{M} c_{ji} = 1, \quad c_{ji} \geq 0, \quad i = 1,2,\cdots,N-1,\ j = 1,2,\cdots,M\ . \tag{16}$$

We now want to use the EM algorithm to derive a new parameter set

$$\hat{\theta} = \{\hat{a}_{ji}(k),\ i = 1,2,\cdots,N-1,\ j = 1,2,\cdots,N; \quad \hat{c}_{ji}(k),\ i = 1,2,\cdots,N-1,\ j = 1,2,\cdots,M\}$$

satisfying constraints (15) and (16) with $a_{ji}$ and $c_{ji}$ replaced by $\hat{a}_{ji}$ and $\hat{c}_{ji}$, respectively. To update $a_{ji}$ to $\hat{a}_{ji}$, we define

$$\tilde{\Lambda}_k = \prod_{l=1}^{k} \prod_{s=1}^{N} \prod_{r=1}^{N-1} \left( \frac{\hat{a}_{sr}(k)}{a_{sr}} \right)^{\langle X_{l-1}, e_r \rangle \langle X_l, e_s \rangle},$$

and set $\left. \frac{dP_{\hat{\theta}}}{dP_{\theta}} \right|_{\mathcal{F}_k} = \tilde{\Lambda}_k$. The above definition of the new probability measure $P_{\hat{\theta}}$ is justified by the following lemma.

**Lemma 3** *Assuming* $X_k = e_r$ ($1 \leq r \leq N-1$), *we have* $E_{\hat{\theta}}[X_{k+1}^s \mid \mathcal{F}_k] = \hat{a}_{sr}(k)$ ($1 \leq s \leq N$), *where* $E_{\hat{\theta}}$ *is the expectation under probability measure* $P_{\hat{\theta}}$.

**Theorem 2** *Based on the EM algorithm, the new estimates for* $a_{sr}$ *given observations up to time* $k$ ($X_{k-1} \neq e_N$ *or* $Z_{k-1} \neq g_{M+1}$) *are given by*

$$\hat{a}_{sr}(k) = \frac{\hat{\mathcal{J}}_k^{rs}}{\hat{\mathcal{O}}_k^{r}} = \frac{\langle \gamma_k(\mathcal{J}_k^{rs}), \mathbf{1} \rangle}{\langle \gamma_k(\mathcal{O}_k^{r}), \mathbf{1} \rangle}.$$

In the next theorem, we obtain the estimates of $c_{sr}$.

**Theorem 3** *Based on the EM algorithm, the new estimates for* $c_{sr}$ *given observations up to time* $k$ ($X_{k-1} \neq e_N$ *or* $Z_{k-1} \neq g_{M+1}$) *are given by*

$$\hat{c}_{sr}(k) = \frac{\hat{\mathcal{T}}_k^{rs}}{\hat{\tilde{\mathcal{O}}}_k^{r}} = \frac{\langle \gamma_k(\mathcal{T}_k^{rs}), \mathbf{1} \rangle}{\langle \gamma_k(\tilde{\mathcal{O}}_k^{r}), \mathbf{1} \rangle}.$$

## *Numerical example*

To illustrate the estimation procedure developed above, we used a real data set obtained from the Cardinal River Coals (CRC) in Canada. It contained information on the working status and oil analysis results for wheel motors. The whole CRC data set consisted of 175 histories. We selected one of them for our analysis. Oil inspection intervals (measured in working age) in the raw data set varied quite a bit. To reduce the variation in inspection intervals to a certain degree, we deleted inspections with short intervals so as to make the assumption of an equal interval between inspections approximately valid. After the modification, the selected history had 97 records of measurement information (results from the analysis of oil samples) and the 98th observation was a failure. Each measurement consisted of 12 data values (levels of metal particles in ppm, sediment level, etc.). We found two of them, Fe (iron) and Sed (sediment) significant by using EXAKT software (EXAKT manual, 2000). To discretize the measurements, we defined a new measurement variable $Y$ with $M = 4$ states: $Y = f_1$ if $\text{Fe} \le \text{mean(Fe)}$ and $\text{Sed} \le \text{mean(Sed)}$ ; $Y = f_2$ if $\text{Fe} \le \text{mean(Fe)}$ and $\text{Sed} > \text{mean(Sed)}$ ; $Y = f_3$ if $\text{Fe} > \text{mean (Fe)}$ and $\text{Sed} \le \text{mean(Sed)}$ ; $Y = f_4$ if $\text{Fe} > \text{mean(Fe)}$ and $\text{Sed} > \text{mean(Sed)}$ . Here, $\text{mean}(\cdot)$ denotes the mean value of a measurement variable in the data set.

Our hidden Markov model was applicable to the modified data set and we applied estimation procedure developed above to the data. We assumed $\text{N} = 3$ states for a wheel motor, the first two are working states and the third one is the failure state. The first state represents good working condition, and the second state indicates that the wheel motor is operating in poor condition. It is reasonable to assume that a new wheel motor is working in good condition, i.e., $X_0 = e_1$ . We calculated the filters of the state and other quantities of interest, and then estimated the transition probability matrices $A$ and $C$ .

The estimation procedure was coded in Matlab. The initial estimates of the model parameters are set as $a_{ji} = \frac{1}{3}, i = 1,2,\ j = 1,2,3;\ c_{ji} = 0.25, i = 1,2, j = 1,2,3,4.$ The Matlab program was run using the modified data set. After 70 iterations, the differences between the present estimates and the previous estimates for $A$ and $C$ were all less than $10^{-5}$ . The calculations took about 2 minutes of CPU time. The final estimates of parameters $A$ and $C$ were:

$$\hat{A} = \begin{pmatrix} 0.9469 & 0.0488 \\ 0.0531 & 0.9270 \\ 0.0000 & 0.0242 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 0.3885 & 0.2022 \\ 0.5337 & 0.0544 \\ 0.0506 & 0.6347 \\ 0.0272 & 0.1087 \end{pmatrix}.$$

From the computed results, we observe that the wheel motor has a very high probability (0.9469) to remain in the good working state at the next inspection epoch if it is in good condition at present. If the wheel motor is in the poor working state, it will have about 5% chance to go back to the good state at the next stage (due to routine maintenance). The probability of failure at the next stage when the motor is in poor condition is 0.0242, whereas it is close to zero when it is in good condition.

## *Summary and conclusions*

In this paper, we have studied the estimation problem for a failure-prone system subject to condition monitoring at discrete points of time. To model the system, we have considered an HMM with observable failure state. We have combined the failure information and the information obtained through condition monitoring and applied the change of measure method to obtain a general recursive filter and the filters for states, transitions, and occupation times. A procedure based on the EM algorithm, suitable for off-line parameter estimation, has been proposed. The procedure has been applied to a real data set (after some modifications) and sensible results have been obtained.

There are certain limitations to our model. First, we have assumed that the inspection intervals are of an equal length, which may not always be true in real data. Second, the range of measurements has been considered to be a finite set which may require some discretization of the observation space before the model can be applied to a real situation. We will relax some of these assumptions in our future work.

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