Rectifying Sampling Inspection by Variables for Assuring Average Outgoing Surplus Quality Loss Limit Indexed by Taguchi’s Loss

Maiko MORITA\textsuperscript{1a}, Yasuhiko TAKEMOTO\textsuperscript{2}, and Ikuo ARIZONO\textsuperscript{1b}

\textsuperscript{1a}Graduate School of Engineering, Osaka Prefecture University 1-1 Gakuen-cho, Nakaku, Sakai, Osaka 599-8531, JAPAN  
\textsuperscript{1b}e-mail: morita@eis.osakafu-u.ac.jp  
\textsuperscript{2}School of Business Administration, University of Hyogo 8-2-1 Gakuennnishimachi, Nishiku, Hyogo 651-2197, JAPAN  
\textsuperscript{1b}e-mail: arizono@eis.osakafu-u.ac.jp

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Abstract

In stead of the quality evaluation based on attribute properties such as the percentage of nonconforming products, Taguchi has proposed the concept of “quality loss” as the quality evaluation based on variable properties. Concretely, Taguchi’s loss function has been defined as a quadratic function based on the departure from the target value of a product. Recently, from the viewpoint of assuring the quality loss proposed by Taguchi, Arizono et al. have proposed a single sampling inspection plan based on operating characteristics. This sampling procedure has been designed for the purpose of assuring both the specified producer’s risk for the lot with an acceptable quality loss limit and the consumer’s risk for the lot with a rejectable quality loss limit. Then, the disposition of the rejected lot has not been specifically declared in the sampling inspection plan mentioned above. While, in the sampling inspection by attribute, the rectifying sampling procedure by attributes, in which the rejected lot in the sampling inspection is thoroughly inspected and nonconforming products in the inspected lot are exchanged for conforming products or repaired, exists as a concept of the sampling inspection. In this sampling procedure, the sampling plans have been designed for the purpose of assuring the percentage of nonconforming products in every lot and the upper limit of the average outgoing quality level as the average outgoing percentage of nonconforming products. Then, in this article, the quality evaluation based on Taguchi’s loss function is adopted instead of the quality evaluation based on the percentage of nonconforming products as lot quality. Then, we propose a rectifying sampling inspection by variable indexed by Taguchi’s loss function for the purpose of assuring the upper limit of the maximum expected surplus loss.

Keywords

Average outgoing surplus quality loss limit, Permissible average outgoing surplus quality loss, Rectifying sampling inspection, Single sampling inspection plan by variables, Taguchi’s loss function.
1. Introduction

Many sampling inspection procedures for the purpose of assuring the quality on lot have been considered for complying with situation in acceptance inspection, in-process inspection and shipping inspection. Furthermore, we can select attribute data or variable data as measurement characteristics in compliance with the quality characteristics, for instance the number of nonconforming products or the product mean, that should be assured.

By the way, in addition to the concept of acceptance and rejection on the quality evaluation such as the number of nonconforming products or the product mean, Taguchi [10, 11] has presented a different approach to quality improvement in which reduction of deviation from target value is the guiding principle. In this approach, any measured quality value \(x\) of a product characteristic \(X\) defined as variable data brings a loss to consumer in general. According to the theory of the Taguchi method, the loss caused by the deviation from its target value is expressed as a quadratic form with respect to the difference between the actual value and the target value. Then, this quality evaluation as the loss is applied in many kinds of decision making about quality improvement [2, 3, 5, 6, 13]. However, there is little idea that the concept of “quality loss” is applied to the process of sampling inspection for assuring the quality of lot. While, one of the authors of this article has presented the variable sampling inspection by applying the concept of quality loss for the purpose of assuring the lot quality [1]. Then, in the case that we have the mean and variance of the lot quality as \(\mu\) and \(\sigma^2\), the expected loss in the Taguchi method can be evaluated as \(k\{\sigma^2 + (\mu - T)^2\}\), where \(k\) denotes the proportional coefficient based on the functional limit of quality and the monetary loss brought by the product which cannot fulfill its function. In the traditional sampling inspection, there isn’t so much of a difference between the concept of the attribute sampling inspection for assuring percentage of nonconforming and the concept of the variable sampling inspection for assuring the mean of product quality characteristics under the known variance. From this concept, it is found that the loss for each lot is not necessarily uniform, even though the lots have the same fraction defective. Hence, from the viewpoint of the Taguchi’s loss criterion, Arizono et al. [1] have proposed a new variable sampling inspection in order to assure the loss caused by the deviation from its target value for a normally distributed quality characteristics and derived a procedure for designing the proposed sampling inspection plan. Then, this sampling inspection plans have assured that the producer’s risk and the consumer’s risk is less than or equal to a given value respectively when the upper of the expected loss on acceptance lot and the lower of expected loss on rejection lot are defined.

However, in general, the variable sampling inspection plan with operating characteristics does not provide the treatment of rejected lot. While, there is a rectifying sampling inspection by attributes to assure the lot quality after inspection [7, 12]. In this rectifying sampling inspection plan, the judgment of acceptance or rejection for the inspected lot is decided through the sampling inspection, and the rejected lot in the sampling inspection is thoroughly inspected. And then, the nonconforming products detected in the total inspection are replaced by the conforming products. In this sampling procedure, the sampling plans have been designed for the purpose of assuring the upper limit of
the average outgoing quality level (the average outgoing quality limit) as the average outgoing percentage of nonconforming products.

In this article, the quality evaluation based on Taguchi’s loss function is adopted instead of the quality evaluation based on the percentage of nonconforming products as lot quality. Therefore, instead of the rectifying sampling inspection by attributes based on the quality evaluation such as the percentages of nonconforming products as the criterion of attribute quality evaluation, we propose the rectifying sampling inspection by variables based on the quality loss as the criterion of variable quality evaluation. We consider the rectifying sampling inspection by variables indexed by Taguchi’s loss function for the purpose of assuring that the upper limit of the maximum expected surplus loss (the average outgoing surplus quality loss limit; AOSQLL) after inspection is less than or equal to the permissible average outgoing surplus quality loss; PAOSQL. In particular, we develop the designing algorithm for determining the criterion of acceptance judgment and the economical sample size. Through the some numerical examples, the effectiveness of the proposed rectifying sampling inspection plan by variables is illustrated and verified.

2. Concept of Assurance for the Average Outgoing Surplus Quality Loss

Suppose that the quality distribution obeys a normal distribution $N(\mu, \sigma^2)$, where $\mu$ and $\sigma^2$ denote the mean and variance, respectively. According to the concept of the Taguchi’s loss, the loss caused by the deviation from the target value of the quality characteristic is expressed as a quadratic from respect to the difference between the actual value $x$ and target value $\mu_0$. Therefore, based on the mean $\mu$ and variance $\sigma^2$ in the actual quality characteristic distribution, the expected loss per a product is specified as

$$E[k(x - \mu_0)^2] = k \left\{ (\mu - \mu_0)^2 + \sigma^2 \right\},$$

(1)

where $k$ denotes the proportional coefficient based on the functional limit of quality. Simply, we denote $k = 1$ without the loss of generality because $k$ is constant. Then, let $(\mu, \sigma^2) = (\mu_0, \sigma_0^2)$ be the mean and variance in the ideal quality characteristic distribution. In this case, from Eq.(1), we have

$$E[(x - \mu_0)^2|(\mu, \sigma^2) = (\mu_0, \sigma_0^2)] = \sigma_0^2 \equiv \tau_0^2.$$

Then, $\tau_0^2$ can be interpreted as the unavoidable loss that exists even in the ideal quality characteristic distribution. While, we define the variance in the ideal lot, the minimum variance that is appropriate to its cost and can realize.

Because we define the variance $\sigma_0^2$ as the variance in the ideal quality characteristic distribution, the variance $\sigma_0^2$ means the feasible minimum variance as a result of balancing with a cost and a technical standard. Therefore, we can assume the relation for the variance on any quality distribution as follow:

$$\sigma^2 \geq \sigma_0^2.$$

(2)

Then, the following relation in any $\tau^2$ under $(\mu, \sigma^2)$ has to be satisfied:

$$\tau^2 = (\mu - \mu_0)^2 + \sigma^2 \geq \tau_0^2(= \sigma_0^2).$$

(3)
Eq. (3) denotes that the expected loss per a product has to be no less than or equal to that in the ideal situation.

Let $Q(\tau^2)$ be the probability that the lot on $\tau_0^2$ is accepted in the variable sampling inspection plan $(n, D)$, where $n$ and $D$ denote the sample size and acceptance value, respectively. Then, we name the probability $1 - Q(\tau_0^2)$ that the lot on $\tau_0^2$ is rejected, the producer’s risk and consider the sampling inspection plan satisfying that this probability is equal to the specified value $\alpha$.

While, with respect to the lot which is rejected in this sampling inspection plan, we inspect totally the products in the rejected lot and the nonconforming products detected in the total inspection are replaced by the conforming products. However, we have to provide a rectifying rule that what quality characteristic value products have is replaced what quality characteristic value products have. In this case, let $\tau_0^2$ be the expected loss on the ideal quality characteristic distribution and treat this loss as the target quality. Accordingly, we consider the rectifying rule satisfying that the target value of the expected loss after rectifying is $\tau_0^2$. Therefore, we plan that the expected loss after the above replacement through the total inspection is made equal to $\tau_0^2$. However, there are many rectifying procedures and some cumbersome, complicated and difficult procedures are included in them. So, we think that these cumbersome procedures are not always appropriate.

Accordingly, in order to assure that the expected loss after rectifying is less than or equal to $\tau_0^2$, we adopt the following rectifying procedure as the most simplest and easiest procedure, that is, we employ the rectifying rule that the products with the expected loss more than $\tau_0^2$ detected in the total inspection are replaced with the product with the expected loss less than $\tau_0^2$. Therefore, the product is shipped directly when the following relation for $x$ is satisfied

$$(x - \mu_0)^2 \leq \tau_0^2 \quad (= \sigma_0^2),$$

where this expression is renewed as follow

$$\mu_0 - \sigma_0 \leq x \leq \mu_0 + \sigma_0. \quad (4)$$

Then, the rectifying rule for the actual value $x$ can be presented as:

$$\begin{cases} 
\text{if } x \text{ satisfy the relation of Eq.}(4), \text{ then ship it}, \\
\text{otherwise, replace it by the product satisfying the relation of Eq.}(4).
\end{cases}$$

Moreover, let $T^2$ be the expected loss per a product on the rectified lot after replacing, it is clearly that $T^2$ is less than or equal to $\tau_0^2$. Similarly, let $L$ denote the expected loss per a product on all lots after rectifying. The following relation is derived

$$L = \tau^2 Q(\tau^2) + T^2 \left\{1 - Q(\tau^2)\right\}. \quad (5)$$

However, it is not easy to obtain the value $L$, because it is difficult to evaluate the value of $T^2$ exactly in advance.

By the way, it is clearly that the value $T^2$ is less than or equal to $\tau_0^2$ from Eq. (4). Therefore, $L$ is the expected loss limit on the lot after rectifying, if $T^2$ is replaced by $\tau_0^2$ on Eq. (5). Let $L_{\text{upper}}$ represent this expected loss limit, then we have

$$L_{\text{upper}} = \tau^2 Q(\tau^2) + \tau_0^2 \left\{1 - Q(\tau^2)\right\}. \quad (6)$$
Moreover, we can obtain the following relation

\[ L_{\text{upper}} - \tau_0^2 = (\tau^2 - \tau_0^2)Q(\tau^2) \equiv S. \tag{7} \]

Then, the value \( S \) on Eq. (7) is interpreted the maximum value of the surplus expected loss more than \( \tau_0^2 \) per a product in ideal quality characteristic distribution (in what follows, we call “average outgoing surplus quality loss (AOSQL”) ). We consider the sampling inspection plan assuring that the value \( S \) is less than or equal to a specified permissible average outgoing surplus quality loss \( S_{\text{limit}} \) (PAOSQL). Consequently, let \( \alpha \) be the specified producer’s risk on the expected loss \( \tau_0^2 \), in this study, we propose the variable sampling inspection plan that \( S \) is less than or equal to \( S_{\text{limit}} \), that is, the algorithm for designing the sampling inspection plan satisfying the following relation:

\[ (\tau^2 - \tau_0^2)Q(\tau^2) \leq S_{\text{limit}}, \tag{8} \]

is developed.

3. Proposal of Design Procedure for the Rectifying Sampling Inspection Plan

Let \( x_j, j = 1, 2, ..., n \), be random samples from a normal distribution \( N(\mu, \sigma^2) \), but \( \mu \) and \( \sigma^2 \) is unknown. Taguchi has proposed the estimator \( \hat{\tau}^2 \) of the expected loss defined by

\[ \hat{\tau}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2 = (\bar{x} - \mu_0)^2 + s^2, \tag{9} \]

where \( \bar{x} \) and \( s^2 \) denote the maximum likelihood estimates of \( \mu \) and \( \sigma^2 \) calculated as follows:

\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \]

\[ s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2. \]

It is known that the static \( n\hat{\tau}^2/\sigma^2 \) obeys a noncentral chi-square distribution with \( n \) degrees of freedom and noncentrality parameter \( n\xi^2 \), where \( \xi^2 \) is defined as

\[ \xi^2 = \frac{(\mu - \mu_0)^2}{\sigma^2}. \tag{10} \]

Then, we think about a static \( \rho \), where \( \rho \) is defined as

\[ \rho = \frac{1 + \xi^2}{1 + 2\xi^2} \frac{n\hat{\tau}^2}{\sigma^2}. \tag{11} \]

Based on the noncentral chi-square distribution with \( n \) degrees of freedom and noncentrality parameter \( n\xi^2 \), the mean and variance of the static \( \rho \) are respectively given by

\[ E[\rho] = \frac{1 + \xi^2}{1 + 2\xi^2} E\left[ \frac{n\hat{\tau}^2}{\sigma^2} \right] = \frac{n(1 + \xi^2)^2}{1 + 2\xi^2}, \]

\[ V[\rho] = \left( \frac{1 + \xi^2}{1 + 2\xi^2} \right)^2 V\left[ \frac{n\hat{\tau}^2}{\sigma^2} \right] = \frac{2n(1 + \xi^2)^2}{1 + 2\xi^2}, \]
and coincide with those of chi-square distribution with $\phi$ degrees of freedom, where

$$\phi = \frac{n(1 + \xi^2)^2}{1 + 2\xi^2}. \quad (12)$$

Accordingly, the chi-square distribution with $\phi$ degrees of freedom can be employed as the approximate distribution of $\rho$.

From Eq.(3) and Eqs.(10)–(12), we have the following relation

$$\rho = \frac{\tau^2}{\tau^2}. \quad (13)$$

Then, by using the Patnaik’s approximation [8], we obtain the approximation model:

$$\tau^2 \sim \frac{\tau^2\chi^2_{\phi}}{\phi}, \quad (14)$$

where $\chi^2_{\phi}$ means the chi-square distribution with $\phi$ degrees of freedom. By making use of this approximation model, we structure the inspection plan. Let $D$ be the acceptance value, it is constructed as

$$D = \begin{cases} \text{if } \tau^2 < D, & \text{then accept the lot,} \\ \text{otherwise,} & \text{reject the lot.} \end{cases} \quad (16)$$

And the rejected lot is inspected totally. When the expected loss on the lot obeying the ideal quality characteristic distribution is $\tau^2_0$, we design the sampling inspection plan that the producer’s risk is less than or equal to the specified value $\alpha$. Then, let us consider two cases, $\tau^2 = \tau^2_0$ and $\tau^2 = \tau^2_1 > \tau^2_0$. In the case of $\tau^2 = \tau^2_0$, since the distribution of the product quality is defined as $N(\mu_0, \sigma^2_0)$, the approximate model is given as

$$\tau^2 \sim \frac{\tau^2\chi^2_{\phi}}{\phi}. \quad (15)$$

Then, $\phi_0$, which is the degrees of freedom on the approximate model, satisfies the relation of $\phi_0 = n$, because of the condition of $\xi^2 = 0$ from Eq.(10). Therefore, the acceptance value $D \equiv \tau^2_0\chi^2_n(\alpha)/n$ is derived as the percentile for the producer’s risk $\alpha$ by the approximate model on Eq.(15), where $\chi^2_n(\alpha)$ denotes the upper 100$\alpha\%$ percentile of the chi-square distribution with $n$ degrees of freedom. Accordingly, the acceptance rule is determined as

$$\text{if } \tau^2 \leq \frac{\chi^2_n(\alpha)}{n}\tau^2_0, \text{ then accept the lot.} \quad (16)$$

In the case of $\tau^2 = \tau^2_1$, the probability that the lot on $\tau^2_1$ is accepted, $Q(\tau^2_1)$, is derived from the condition that AOSQLL is less than or equal to PAOSQL $S_{\text{limit}}$ as follows:

$$Q(\tau^2_1) \leq \frac{S_{\text{limit}}}{\tau^2_1 - \tau^2_0}. \quad (17)$$

For any pairs of $(\mu_1, \sigma^2_1)$ satisfying the relation of $\tau^2_1 = \sigma^2_1 + (\mu_1 - \mu_0)^2$, there are innumerable pairs of $(\mu_1, \sigma^2_1)$ giving the same value of $\tau^2_1$. Then, because $\tau^2_1$ is a function for
\((\mu_1, \sigma_1^2)\), \(Q(\tau_1^2)\) is a function for \((\mu_1, \sigma_1^2)\). Furthermore, in order to maximize \(Q(\tau_1^2)\), the following formula must be satisfied under the condition Eq.(17):

\[
\max_{(\mu_1, \sigma_1^2) \in \tau_1^2} Q(\tau_1^2) \leq \frac{S_{\text{limit}}}{\tau_1^2 - \tau_0^2}.
\]  (18)

Let \(\phi_1\) be the degree of freedom from the parameter \(\xi_1^2\) calculated on Eq.(10) applying the pair of \((\mu_1, \sigma_1^2)\) satisfied the expected loss \(\tau_1^2\), then we obtain the approximate model:

\[
\tau_1^2 \sim \frac{\tau_1^2 \chi_1^2}{\phi_1},
\]  (19)

where

\[
\phi_1 = \frac{n(1 + \xi_1^2)^2}{1 + 2\xi_1^2}, \xi_1^2 = \frac{(\mu_1 - \mu_0)^2}{\sigma_1^2}.
\]

Considering the probability that the lot on \(\tau_1^2\) is rejected under Eq.(16), the following relation has to be satisfied

\[
\min_{(\mu_1, \sigma_1^2) \in \tau_1^2} \Pr \left( \frac{\tau_1^2}{\tau_1^2 - \tau_0^2} \geq \frac{\chi_n^2(\alpha)}{n\phi_1} \right) \geq 1 - \frac{S_{\text{limit}}}{\tau_1^2 - \tau_0^2}.
\]

Therefore, we can obtain the relation

\[
\frac{\chi_n^2(\alpha)}{n\tau_0^2} \leq \min_{(\mu_1, \sigma_1^2) \in \tau_1^2} \frac{\chi_1^2(\epsilon)}{\phi_1},
\]  (20)

where

\[
\epsilon \equiv 1 - \frac{S_{\text{limit}}}{\tau_1^2 - \tau_0^2}.
\]

In this case, by applying the Willson-Hilferty approximation [4], we consider the behavior of \(\chi_1^2(\epsilon)/\phi_1\) for \(\phi\), that is, a pair of \((\mu_1, \sigma_1^2)\) in minimizing the right side of Eq.(20). Next, in order to obtain a pair of \((\mu_1, \sigma_1^2)\) in minimizing the right side of Eq.(20), we have further three sub-cases as follows:

a) \(\tau_1^2 \leq \tau_0^2 + S_{\text{limit}}\)

It means \(\epsilon \leq 0\). Then, because the expected loss is always less than or equal to \(S_{\text{limit}}\), it isn’t necessary to consider this sub-case as a condition of designing the sampling inspection plan.

b) \(\tau_0^2 + S_{\text{limit}} < \tau_1^2 \leq \tau_0^2 + 2S_{\text{limit}}\)

Then, \(0 < \epsilon \leq 0.5\). Based on Willson-Hilferty approximation, this case can correspond \(0 \leq u_\epsilon < \infty\), where \(u_{1-\theta}\) represent the upper \(100(1-\theta)\%\) percentile of a standard normal distribution. Then, we consider the function \(\chi_1^2(\epsilon)/\phi_1\) in the range

\[
\frac{8}{9n} \leq u_\epsilon < \infty.
\]
This function $\chi^2_{\phi_1}(\varepsilon)/\phi_1$ is convex upward function. While, the function $\chi^2_{\phi_1}(\varepsilon)/\phi_1$ is monotonous decreasing function for $\phi_1$ (See Appendix A). Therefore, we must have further two conditions as follows, where $\gamma$ is defined by the following relational equation

$$u_{\gamma} = \frac{\sqrt{8}}{9n}.$$  

b-i) $\tau^2_0 + S_{\text{limit}} < \tau^2_1 \leq \tau^2_0 + \frac{S_{\text{limit}}}{\phi_1}$

Then, $0 < \varepsilon \leq 1 - \gamma$. Under this condition, from Appendix A, $\chi^2_{\phi_1}(\varepsilon)/\phi_1$ is monotonous decreasing function for $\phi_1$. Therefore, $\chi^2_{\phi_1}(\varepsilon)/\phi_1$ is a minimum value when $\phi_1$ has a maximum value in the range $\phi_1 \geq n$. In this case, $\phi_1$ is monotonous increasing function for $\xi_1$ and $\xi_1$ is expressed by Eq. (12) as follows:

$$\xi^2_1 = \frac{\tau^2_1}{\sigma^2_1} - 1.$$  

Under the fixed value of $\tau^2_1$, $\xi_1$ has a maximum value in the condition of $\sigma^2_1 = \sigma^2_0$. Therefore, when the parameter $\xi^2_1$ is defined by the following pair of $(\mu_1, \sigma^2_1)$ under the fixed value of $\tau^2_1$:

$$(\mu_1, \sigma^2_1) = (\mu_0 \pm \sqrt{\tau^2_0 - \sigma^2_0}, \sigma^2_0).$$

Then, $\phi_1$ becomes maximum value $\phi_{1\max}$, furthermore $\chi^2_{\phi_{1\max}}(\varepsilon)/\phi_{1\max}$ takes a minimum value. Then, the following condition equation regarding sample size $n$ is derived from Eq. (20):

$$\frac{\tau^2_0}{\tau^2_1} \leq \frac{n}{\chi^2_n(\alpha)} \frac{\chi^2_{\phi_{1\max}}(\varepsilon)}{\phi_{1\max}}.$$  

(21)

b-ii) $\tau^2_0 + S_{\text{limit}} < \tau^2_1 \leq \tau^2_0 + 2S_{\text{limit}}$

Then, $0 < \varepsilon \leq 1 - \gamma$. Under this condition, from Appendix A, $\chi^2_{\phi_1}(\varepsilon)/\phi_1$ is convex upward function. Then, let $\phi_{1\min}$ and $\phi_{1\max}$ be the minimum value and maximum value for $\phi_1$, respectively. Then, $\chi^2_{\phi_1}(\varepsilon)/\phi_1$ takes the minimum value when $\phi_1$ is either $\phi_{1\min}$ or $\phi_{1\max}$. Since $\phi_1$ is a monotonous increasing function for $\xi_1$ ($\xi^2_1 \geq 0$), under the condition of $\xi^2_1 = 0$:

$$(\mu_1, \sigma^2_1) = (\mu_0, \tau^2_1),$$

$\phi_1$ takes the minimum value $\phi_{1\min} = n$. While, $\phi_{1\max}$ is obtained under the condition of $\sigma^2_1 = \sigma^2_0$:

$$(\mu_1, \sigma^2_1) = (\mu_0 \pm \sqrt{\tau^2_0 - \sigma^2_0}, \sigma^2_0).$$

Consequently, the following condition equation regarding sample size $n$ is derived

$$\frac{\tau^2_0}{\tau^2_1} \leq \frac{n}{\chi^2_n(\alpha)} \min \left( \frac{\chi^2_{\phi_{1\max}}(\varepsilon)}{\phi_{1\max}}, \frac{\chi^2_n(\varepsilon)}{\phi_{1\max}} \right).$$  

(22)

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c) \( \tau_0^2 + 2S_{\text{limit}} < \tau_1^2 \)

Then, it is obvious that \( 0.5 < \varepsilon \). Under this condition, \( \chi_{\phi_1}(\varepsilon)/\phi_1 \) is monotonous increasing function for \( \phi_1 \), from Appendix B. Therefore, \( \chi_{\phi_1}(\varepsilon)/\phi_1 \) is minimized in the range \( \phi_1 \geq n \) under the condition

\[
(\mu_1, \sigma_1^2) = (\mu_0, \tau_1^2).
\]

Then, we have \( \phi_1 = \phi_{\min} = n \), and the following condition equation regarding sample size \( n \) is derived

\[
\frac{\tau_0^2}{\tau_1^2} \leq \frac{n}{\chi_n(\alpha)} \frac{\chi_n^2(\varepsilon)}{\chi_n^2(\alpha)} = \frac{\chi_n^2(\varepsilon)}{\chi_n^2(\alpha)}.
\] (23)

As mentioned previously, the purpose of this article is to provide the design procedure for the sampling inspection plan that \( S \) is less than or equal to \( S_{\text{limit}} \) for any \( \tau_1^2 (> \tau_0^2) \). Therefore, we adopt the maximum value in a thing of the sample size obtained by using Eqs.(21)-(23). In this case, let the adopted sample size denote \( n_{\max} \). we can obtain the following decision rule:

\[
\text{if } \frac{\tau_0^2}{\tau_1^2} \leq \frac{\chi_n^2(\alpha)}{n_{\max}} = D, \text{ the lot is accepted.}
\] (24)

By the above, we can construct the sampling inspection plan for assuring that AOSQLL is less than or equal to PAOSQL under the given producer’s risk \( \alpha \) on \( \tau_1^2 \).

4. Numerical Examples

In order to illustrate the validity of the provided sampling inspection, consider the some numerical examples. Let \( N(\mu_0, \sigma_0^2) = N(0.0, 1.0) \), and then the ideal expected loss is given as \( \tau_0^2 = 1.0 \). Then, let \( \alpha = 0.05 \) and \( S_{\text{limit}} = 0.25 \). Figure 1 shows the required sample size for each of the expected loss \( \tau_1^2 \). In this case, we can obtain sample size \( n_{\max} = 30 \) and the acceptance value \( D = 1.459 \) for satisfying that AOSQLL is less than or equal to a given value PAOSQL. Similarly, Figure 2 shows the required sample size for each of the expected loss \( \tau_1^2 \) in the case of \( \alpha = 0.05 \) and \( S_{\text{limit}} = 0.35 \). Furthermore, the required sample size for each of the expected loss \( \tau_1^2 \) in the case of \( \alpha = 0.01 \) and \( S_{\text{limit}} = 0.25 \) is illustrated by Figure 3. In these cases, we can obtain sample size \( n_{\max} = 17 \) and \( n_{\max} = 63 \), respectively. We can confirm that \( n_{\max} \) in Figure 2 is smaller than that in Figure 1 because the expected loss in Figure 2 larger than that in Figure 1. On the other hand, we can confirm that \( n_{\max} \) in Figure 3 is larger than that in Figure 1 because a producer’s risk in Figure 3 is smaller than that in Figure 1.

Just to make sure, we verify that the AOSQLL for each \( \tau_1^2 \) is satisfied by applying the proposed inspection plan \( (n_{\max}, D) \). The approximate model \( \tau_1^2 \) on \( \tau_1^2 \) is expressed as Eq.(19). For a pair of \( (\mu_1, \sigma_1^2) \) satisfied \( \tau_1^2 \), let \( S^* \) be AOSQLL for each \( \tau_1^2 \) realized by proposed inspection plan. Then \( S^* \) is satisfied under the equal condition of Eq.(20) as follows:

\[
\frac{\chi_{\phi_0}(\alpha)}{n_{\max}} \frac{\tau_0^2}{\tau_1^2} = \frac{\chi_{\phi_1}(\varepsilon^*)}{\phi_1} \frac{\tau_0^2}{\tau_1^2},
\]

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where
\[ \varepsilon^* = 1 - \frac{S^*}{\tau_1^2 - \tau_0^2}. \]

Therefore, we obtain the following relation:
\[ \chi_{\phi_1}^2(\varepsilon^*) = \frac{\phi_1 \tau_0^2}{n_{\max} \tau_1^2} \chi_{\phi_0}^2(\alpha). \]

Furthermore, we can obtain the upper 100\( \varepsilon^*\)% percentile of based on Wilson-Hilferty approximation [4] as follows
\[ u_{\varepsilon^*} = \sqrt{\frac{9\phi_1}{2} \left( \frac{\chi_{\phi_1}^2(\varepsilon^*)}{\phi_1} + \frac{2}{9\phi_1} - 1 \right)}. \]

In this case, we can evaluate the AOSQLL \( S^* \) for each \( \tau_1^2 \) realized by the proposed inspection plan as
\[ S^* = (\tau_1^2 - \tau_0^2)(1 - \Phi(u_{\varepsilon^*})), \tag{25} \]

where \( \Phi(u) \) denotes the distribution function of the standard normal distribution. Then, in Figures 4-6, the AOSQLL \( S^* \) for each \( \tau_1^2 \) realized by proposed inspection plan from Eq.(25) is illustrated. The setting condition of Figures 4-6 comply with that of Figures 1-3, respectively. In Figures 4-6, we can find that the AOSQLL \( S^* \) is less than or equal to the PAOSQL \( S_{\text{limit}} \) on the inspection plan for each \( n_{\max} \).

5. Concluding Remarks

In this study, we have propose the rectifying sampling inspection to assure that the AOSQLL based on Taguchi’s loss function is always less than or equal to a specified value PAOSQL, and provided the design procedure for this inspection plan. Then, there are innumerable pairs of the mean and variance of the expected loss of product quality which is larger than that realized in the ideal situation. We have considered the sampling inspection plan that the expected loss is less than or equal to a specified value under the worst situation. Therefore, according to the proposed inspection plan, we can assure that the expected loss is less than or equal to a specified value for the lot of every pairs of the mean and the variance. Moreover, we verify to assure that the AOSQLL is less than or equal to the PAOSQL through some numerical examples.

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Appendix A: Behavior of $\chi_{\phi_1}(\varepsilon)/\phi_1$ for $\phi_1$ in $\tau_0^2 + S_{\text{limit}} \leq \tau_1^2 < \tau_0^2 + 2S_{\text{limit}}$

Employ the Wilson–Hilferty approximation for the upper percentile of $\chi^2$ distribution:

$$\chi_{\phi_1}(1 - \theta) = \phi_1 \left\{ 1 - \frac{2}{9\phi_1} + u_{1-\theta}\sqrt{\frac{2}{9\phi_1}} \right\}^3,$$

where $u_{1-\theta}$ for $\theta(\geq 0.5)$ denotes the upper 100$(1 - \theta)$ percentile of the standard normal distribution. We consider the function:

$$\frac{\chi_{\phi_1}(1 - \theta)}{\phi_1} = \left\{ 1 - \frac{2}{9\phi_1} + u_{1-\theta}\sqrt{\frac{2}{9\phi_1}} \right\}^3.$$

Then, we have

$$\frac{d}{d\phi_1} \frac{\chi_{\phi_1}(1 - \theta)}{\phi_1} = \frac{\sqrt{2}}{6\phi_1^3} \left\{ \sqrt{\frac{8}{9\phi_1}} - u_{1-\theta} \right\} \left\{ 1 - \frac{2}{9\phi_1} + u_{1-\theta}\sqrt{\frac{2}{9\phi_1}} \right\}^2.$$

Further, since $\phi_1 \geq n$, in the case that $u_{1-\theta} > \sqrt{8/(9n)}$, it is obvious that

$$\sqrt{\frac{8}{9\phi_1}} - u_{1-\theta} < 0.$$

Then, we know that the function $\chi_{\phi_1}(1 - \theta)/\phi_1$ is a monotonous decreasing function for $\phi_1$. On the other hand, in the case that $u_{1-\theta} \leq \sqrt{8/(9n)}$, we know that the function $\chi_{\phi_1}(1 - \theta)/\phi_1$ is the convex upward function for $\phi_1$.

Appendix B: Behavior of $\chi_{\phi_1}(\varepsilon)/\phi_1$ for $\phi_1$ in $\tau_0^2 + 2S_{\text{limit}} \leq \tau_1^2$

Based on the approximation of the upper percentile $\chi_{\phi_1}(1 - \theta)$ for $\theta < 0.5$. We have also

$$\frac{\chi_{\phi_1}(1 - \theta)}{\phi_1} = \left\{ 1 - \frac{2}{9\phi_1} - u_{\theta}\sqrt{\frac{2}{9\phi_1}} \right\}^3.$$

Furthermore, the differential coefficient for $\phi_1$ is derived as

$$\frac{d}{d\phi_1} \frac{\chi_{\phi_1}(1 - \theta)}{\phi_1} = \frac{\sqrt{2}}{6\phi_1^3} \left\{ \sqrt{\frac{8}{9\phi_1}} + u_{1-\theta} \right\} \left\{ 1 - \frac{2}{9\phi_1} - u_{\theta}\sqrt{\frac{2}{9\phi_1}} \right\}^2.$$

Since $\phi_1, u_{\theta} > 0$, it is obvious that

$$\frac{d}{d\phi_1} \frac{\chi_{\phi_1}(1 - \theta)}{\phi_1} > 0.$$

Therefore, we know that the function $\chi_{\phi_1}(1 - \theta)/\phi_1$ is a monotonous increasing function for $\phi_1$. 

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References


Figure 1: the required sample size for each of the expected loss $\tau_1^2$ ($S_{\text{limit}} = 0.25, \alpha = 0.05$)

Figure 2: the required sample size for each of the expected loss $\tau_1^2$ ($S_{\text{limit}} = 0.35, \alpha = 0.05$)

Figure 3: the required sample size for each of the expected loss $\tau_1^2$ ($S_{\text{limit}} = 0.25, \alpha = 0.01$)
Figure 4: AOSQL for $\tau_1^2$ under $n_{\text{max}}$ $(S_{\text{limit}} = 0.25, \alpha = 0.05)$

Figure 5: AOSQL for $\tau_1^2$ under $n_{\text{max}}$ $(S_{\text{limit}} = 0.35, \alpha = 0.05)$

Figure 6: AOSQL for $\tau_1^2$ under $n_{\text{max}}$ $(S_{\text{limit}} = 0.25, \alpha = 0.01)$