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Using “Last-Minute” Sales for Vertical Differentiation on the Internet

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Abstract

In Internet based commerce, sellers often use multiple distribution channels for the sale of standard consumer goods. We study a model of second degree price discrimination in which a monopolist sells to risk-averse buyers. The seller uses two channels that differ in their risk attributes. In one channel prices and qualities are fixed and availability is assured. In the second channel, the seller offers a joint-distribution of prices and qualities and may not guarantee availability. We characterize optimal two-channel selling policies. We show that it can be optimal to offer multiple identical items in a random sale event. However, the seller cannot benefit by offering two distinct quality levels in a sale event that is held with a probability less than one.

JEL Classifications: D42, L1
1. Introduction

The Internet has seen a proliferation in the use of last-minute promotions by sellers of various products and services. For example, through its e-Saver™ program US Airways offers on its web-site each week discounted fares for travel to selected destinations on the same weekend. A possible explanation for such behavior is that sellers such as US Airways are deliberately embedding price and availability uncertainties into their sales channels in order to employ second degree price discrimination among buyers who are risk averse. Buyers who assign higher values to the offered products are typically more reluctant to risk compromising their surplus and are therefore prone towards early “risk free” purchase at higher posted prices while buyers with lower values may be willing to wait and attempt to acquire products at bargain prices.

According to our observations, US Airways offers special e-Saver fares only on travel in economy-class. However, some retail travel specialists such as lastminute.com and site59.com offer also high-end hotel accommodations and luxury cruise vacations in “last-minute” promotions. Can US Airways benefit by offering special last-minute promotions on travel in business-class, and if so, how?

This paper provides a simple framework that permits the investigation of these issues. The analysis provides insights into the determination of profit-maximizing selling policies that involve the offering of products of distinct qualities though a mechanism that involves both advanced-purchase transactions and random “last-minute” sales.

A substantial body of literature examines the provision of quality by a discriminating monopolist. Of particular relevance to our discussion are works by Mussa and Rosen (1978), Deneckere and McAfee (1996), Johnson and Myatt (2003), and Anderson and Dana (2005). A basic result that follows from all of these models is that when products’ quality is endogenous and buyers self-select among quality-levels a discriminating monopolist will distort the provision of quality at the lower end of the quality spectrum. Interestingly, our theoretical results indicate that when buyers are risk averse the use of random “last-minute” sales can either alleviate or exacerbate such quality distortions, depending on the circumstances. Nonetheless, we show that when the degree of buyers’ risk aversion exceeds a threshold, a monopolist who uses “last-minute” sales can maximize her profits by offering only products of efficient quality.

The potential segmentation benefits that may arise from incentive schemes yielding random outcomes have long been recognized. In two independent seminal studies, Matthews (1983) and
Maskin and Riley (1984) characterize optimal auctions with risk-averse buyers under different sets of assumptions. While assuming that buyers have uniform utility functions and differ only in their valuation of a single item, both studies establish that the seller can devise a truth-revelation mechanism that strictly dominates any one-price scheme while inducing an equilibrium in which almost all buyers are faced with risk. With such an “optimal auction” every buyer is induced to reveal his value of the good; he is then assigned a schedule that includes a “bid submission” fee, a probability of winning the item, and an “acquisition price” to be paid only if the item is won. In a related study, Marom and Seidmann (2005) characterize an optimal scheme for the sale of multiple identical items by a monopolist in a market comprising risk-averse buyers. They assume that the monopolist is restricted to the offering of a two-item menu: (i) a unit of the good that is supplied with certainty at a price $p_1$; (ii) a unit of the good that is supplied at a price $y$ that is a random variable drawn from a probability distribution $F$. The authors show that it is optimal for the monopolist to use a distribution $F$ of a “two-point” form; this discrete distribution assigns a probability $\alpha$ to a realization in which a “sale” price $p_2 < p_1$ is offered, and a complementary probability $1 - \alpha$ to a realization in which the price $y$ is high enough so that no buyer is willing to transact in it.

In this paper we extend the modeling framework used in Marom and Seidmann (2005) by considering two products of distinct qualities that are offered by a monopolist. Buyers are risk-averse and differ in their appreciations of quality. We investigate how differences in buyers’ willingness to pay, and the monopolist’s costs of production for each product may affect the equilibrium outcome in this case. In addition, we study the effects of different levels of buyers’ risk aversion on the monopolist’s optimal selling policy. Our theoretical results indicate that it can be optimal to offer the high-quality product alone in a random sale event. It can also be optimal to offer the low-quality product alone in a sale. However, the seller cannot benefit by offering both high and low-quality products (with positive probabilities $\alpha$ and $\beta$, respectively) in a sale that is held with a probability less than one (i.e., when $\alpha + \beta < 1$).

The remainder of the discussion is organized as follows. In §2 we introduce the model. In §2.1 we investigate optimal strategies where the seller offers only one product of standard quality. In §2.2 we study a case where the seller offers two products but does not use randomization. In §2.3 we develop the optimal solution for the general case where the seller offers two products and at the same time can use randomization. Section 3 includes a few concluding remarks.
2. Model

A monopolist sells two distinct products: a high-quality product A and a low-quality product B. The per-unit marginal production cost of product B is fixed and normalized at zero; the per-unit marginal production cost of product A is a positive constant $c$.

Customers are divided equally into two types H and L. A type-$i$ customer’s demand $(i \in \{H, L\})$ is either for one unit of product A, one unit of product B, or for no product at all. The reservation values of type-H and type-L customers for a unit of product B are denoted by $V_H$ and $V_L$ respectively ($V_H \geq V_L$). The corresponding reservation values for product A are $V_H + \delta_H$ and $V_L + \delta_L$.

We make the following model assumptions:

A1). $\delta_H \geq \delta_L > 0$.
A2). $\delta_L \geq c > 0$.

This implies that if the monopolist were serving only the low types she would find it more profitable to sell them high quality. So the high-quality product A is efficient for both buyer types H and L.

Buyers care only about product and price and are completely indifferent as to the time of purchase. The utility function $U_i$ $(i \in \{H, L\})$ is defined as follows.

If a type-$i$ customer buys A: $U_i = (V_i + \delta_i - p_A)^{1-\rho}$, $V_i + \delta_i - p_A \geq 0$.
If a type-$i$ customer buys B: $U_i = (V_i - p_B)^{1-\rho}$, $V_i - p_B \geq 0$.
If a type-$i$ customer buys no product at all: $U_i = 0$.

Customers of both types thus exhibit the same degree of constant relative risk aversion $\rho$ $(0 \leq \rho < 1)$.

Suppose that the seller initially offers the good $j$ $(j \in \{A, B\})$ at price $p_{1j}$. If potential demand and production capacity are not totally exhausted, it pays for the seller to continue and then offer additional units at a lower price $p_{2j} < p_{1j}$, targeting potential buyers that have so far chosen not to buy at $p_{1j}$. Since buyers understand this, they anticipate the subsequent discount, and optimally wait for it. In the absence of a time deadline, similar considerations apply indefinitely. The seller's power to attain strictly positive profits depends on the possibility of a commitment, whereby some potentially profitable future decisions are inhibited. The ex-post opportunity loss (which appears to violate subgame perfection) is amply justified by the ex-ante profitability of the initial commitment.
With any non-degenerate pure strategy, all sales take place at the same time period. Recognizing
the possibility (indeed inevitability) of commitment, we consider an alternative "two-period" selling
strategy that involves randomization. We assume that the seller can commit to a strategy of the form
\[ S = (p_{1A}, p_{1B}, p_{2A}, p_{2B}; \alpha, \beta, \gamma) \]
where:
- \( p_{kj} > 0 \) is the price of product \( j \) in period \( k \) (\( k = 1,2 \)).
- \( 0 \leq \alpha \leq 1 \) is the probability that product A alone is offered in period 2 (at price \( p_{2A} \)).
- \( 0 \leq \beta \leq 1 - \alpha \) is the probability that product B alone is offered in period 2 (at price \( p_{2B} \)).
- \( 0 \leq \gamma \leq 1 - \alpha - \beta \) is the probability that both products are offered in period 2 (at prices \( p_{2A}, p_{2B} \)).

In what follows, our main goal is to solve for the optimal selling strategy \( S \).

2.1 Randomization with a single quality level

In this section we consider a case where only one product is offered, without loss of generality let
us assume that it is product B. In period 1, the price of product B is denoted by \( p_{1B} \). The seller
commits to holding a second-period sale (at price \( p_{2B} \)) with probability \( \beta \). To save on notations we
represent the seller’s strategy \( S \) in this case by \( S = (p_{1B}, p_{2B}, \beta) \). In the corresponding equilibrium,
the seller’s profit is \( \Pi = \Pi(S) \).

If it is optimal for type-H customers to buy in period 2 then the same is true for type-L customers
(see lemma 1). If at the corresponding equilibrium both customers purchase in the same period the
seller can trivially attain or exceed \( \Pi(S) \) with some strategy \( S' \) where \( \beta = 1 \) and \( p_{1B} = p_{2B} \). Hence it
suffices to consider the case where one type (i.e. H) buys in period 1 and the other buys in period 2.
The seller’s profit-maximization problem is thus equivalent to:

\[
\begin{align*}
\max_S & \quad p_{1B} + \beta \cdot p_{2B} \\
\text{s.t.:} & \quad p_{1B} \leq V_H. \\
& \quad p_{1B} \leq V_H - \beta^{1-\rho}(V_H - p_{2B}). \\
& \quad p_{2B} \leq V_L. \\
& \quad p_{1B} \geq V_L - \beta^{1-\rho}(V_L - p_{2B}).
\end{align*}
\]
\[ 0 \leq \beta \leq 1. \]  

Conditions (1.1) and (1.3) are *individual-rationality* constraints for type H and type L customers, respectively. Condition (1.1) means that a type-H customer prefers to purchase product B in the first period to not purchasing any product at all. Condition (1.3) implies that a type-L customer prefers to purchase product B whenever a sale is held at the second period to not purchasing at all. Conditions (1.2) and (1.4) are *incentive-compatibility* constraints for the two customer types. Condition (1.2) implies that a type-H customer prefers to purchase product B in the first period to deferring his purchasing decision to the second period. Condition (1.4) implies that the opposite holds true for a type-L customer.

With no loss of optimality we can restrict our attention to strategies that induce type-L customers to buy with positive probability \( \beta > 0 \) (see theorem 1). It can be shown that a (unique) optimal solution \( S^\ast \) with \( \beta^\ast < 1 \) exists if and only if

\[
\lim_{\beta \to 1} \frac{d\left(V_H - \beta^{1-\rho}(V_H - V_L) + \beta V_L\right)}{d\beta} < 0. 
\]  

(7)

Whereas the above condition holds true if and only if

\[
\rho > \frac{2V_L - V_H}{V_L}. 
\]  

(8)

Whenever condition (3) is satisfied, the optimal point \( S^\ast \) involves:

\[
p_{1B}^\ast = V_H - \beta^{1-\rho}(V_H - p_{2B}) \quad , \quad p_{2B}^\ast = V_L.
\]  

(9)

\[
\beta^\ast = \left(\frac{(1-\rho)V_L}{V_H - V_L}\right)^\frac{1}{\rho} \quad V_H \geq V_L \geq 0, \quad 0 < \rho \leq 1.
\]  

(10)

The above expression for \( \beta^\ast \) is an increasing function of risk-aversion\(^1\) (\( \rho \)) and a decreasing function of the value \( V_H \). We explain the later result as follows. When the difference between \( V_H \) and \( V_L \) increases, type-H customers observe a period 2 price \( (p^\ast_{2B} = V_L) \) that is lower relative to their own value \( (V_H) \). As a result, they have a greater incentive to defer their purchase to period 2. The seller’s best response in this case is to reduce the probability \( \beta \) so as to keep type-H customers indifferent between the two alternative channels.

\(^1\) An explanation of this result is given in Marom and Seidmann (2005).
Whenever condition (3) is not satisfied any strategy that results in selling B at a price $V_L$ to both buyers with probability 1 is optimal.

2.2 Vertical differentiation without randomization

In this section we consider a case where the seller offers two products but does not use randomization. For convenience, let us assume that all sales take place in period 1.

Obviously, only three strategies need be considered by the seller:

- $S_I$ — (“Sell product A to type H and no product to type L”): $p_A = V_H + \delta_H$, $p_B > V_H$.
- $S_{II}$ — (“Sell product A to both types H and L”): $p_A = V_L + \delta_L$, $p_B > V_H$.
- $S_{III}$ — (“Sell product A to type H and product B to type L”): $p_A = V_L + \delta_H$, $p_B = V_L$.

Let $\pi_i$ represent the equilibrium profit that corresponds to strategy $S_i$, $(i \in I, II, III)$. We have:

- $\pi_I \geq \pi_H$ if and only if, $V_H + \delta_H \geq 2(V_L + \delta_L) - c$.
- $\pi_I \geq \pi_{II}$ if and only if, $V_H \geq 2V_L$.
- $\pi_{III} \geq \pi_{II}$ if and only if, $\delta_H \geq 2\delta_L - c$.

2.3 Vertical differentiation with randomization

This section studies optimal selling policies that involve both vertical-differentiation and randomization. The seller’s strategy space consists of all strategies of the form,

$$S = (p_{1A}, p_{1B}, p_{2A}, p_{2B}; \alpha, \beta, \gamma).$$

Lemma 1: Without loss of optimality, the seller can restrict herself to strategies that induce customers of type H to buy in period 1, and customers of type L to buy in period 2.

All proofs are contained in Appendix I.

Corollary: There always exists an optimal strategy with $p_{1B} > V_H$, $p_{2A} = V_L + \delta_L$ and $p_{2B} = V_L$.

Lemma 2: Without loss of optimality, the seller can restrict herself to strategies with $\gamma = 0$.

Theorem 1: In equilibrium, customers of type H buy (product A) with certainty whereas customers of type L buy (either product A or product B) with a non-zero probability.

Notably, theorem 1’s assertion does not generally hold true in the pure vertical-segmentation case (§2.2). Our analysis thus indicates that a monopolist that uses randomization excludes fewer buyers from consumption with certainty\(^2\). The next theorem contains our main result.

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\(^2\) A similar result was obtained in Marom and Seidmann (2005).
**Theorem 2:** Without loss of optimality, the seller can restrict herself to strategies with either 
\[ \alpha = 0, \beta = 0 \] or 
\[ 1 - \alpha - \beta = 0. \]
Simply put, the firm cannot increase its profits by offering more than one product with positive probability in a random (i.e., \( \alpha + \beta < 1 \)) sale event.

**Optimal Selling Strategies**
- Under what circumstances should the monopolist use randomization, and how?

In this subsection we attempt to describe the seller’s optimal strategy choices in various model settings and explain how changes in the model’s parameters can affect them.

Based on our previous results we restrict our attention to strategies 
\[ S = (p_{1A}, p_{1B}, p_{2A}, p_{2B}; \alpha, \beta, \gamma) \] (with \( p_{1B} > V_H, p_{2A} = V_L + \delta_L, p_{2B} = V_L \) and \( \gamma = 0 \)) that belong to (at least) one of the following five categories \( (\Sigma_I, \Sigma_{II}, \Sigma_1, \Sigma_2, \Sigma_3) \); where,

- \( S \in \Sigma_I \) if \( \alpha = 1 \) and \( \beta = 0 \).
- \( S \in \Sigma_{II} \) if \( \alpha = 0 \) and \( \beta = 1 \).
- \( S \in \Sigma_1 \) if \( 0 \leq \alpha \leq 1 \) and \( \beta = 0 \).
- \( S \in \Sigma_2 \) if \( \alpha = 0 \) and \( 0 \leq \beta \leq 1 \).
- \( S \in \Sigma_3 \) if \( 0 \leq \alpha \leq 1 \) and \( \beta = 1 - \alpha \).

For all \( i \in \{II, III, 1, 2, 3\} \) let \( s_i \) be defined as the maximal-profit strategy \( S \) such that \( S \in \Sigma_i \):

\[ \pi_i = \Pi(s_i) = \max_{S \in \Sigma_i} \Pi(S). \] (11)

Strategies that belong to \( \Sigma_{II} \) and \( \Sigma_{III} \) do not involve randomization (see also §2.2). We have,

- \( s_{II} = (p_{1A}^{II}, p_{2A}^{II}) = (V_L + \delta_L, V_L + \delta_L) \) and
- \( s_{III} = (p_{1A}^{III}, p_{2B}^{III}) = (V_L + \delta_H, V_L) \).

Strategies that belong to \( \Sigma_1 \) involve randomized selling of product A in period 2. We use the abbreviated\(^3\) form \( s_1 = (p_{1A}, p_{2A}, \alpha) \). The corresponding profit function is

\[ \pi_1 = p_{1A} + \alpha(V_L + \delta_L) - (1 + \alpha)c. \] (12)

We find that \( \pi_1 \) exceeds \( \pi_{II} \) (i.e. \( 0 < \alpha < 1 \)) if and only if buyers are sufficiently risk averse

\[ \rho > \frac{2(V_L + \delta_L) - V_H - \delta_H - c}{V_L + \delta_L - c}, 0 < \rho < 1. \] (13)

When \( 0 < \alpha < 1 \) we have

\[^3\] \( s_1 = (p_{1A}, p_{1B}, p_{2A}, p_{2B}; \alpha, \beta, \gamma) \) with \( p_{1B} > V_H, \beta = 0 \) and \( \gamma = 0 \).
\[ 1 \alpha = \left( \frac{(1 - \rho)(V_L + \delta_L - c)}{V_H + \delta_H - V_L - \delta_L} \right)^{1-\rho}, 0 < \rho < 1. \] (14)

\[ 1 p_{1A} = V_H + \delta_H - \mu \frac{1-\rho}{\rho} (V_H + \delta_H - V_L - \delta_L); \quad 1 p_{2A} = V_L + \delta_L. \] (15)

Strategies belonging to \( \Sigma_2 \) involve randomized selling of product B in period 2. Let \( s_2 = (2 p_{1A}, 2 p_{2B}, 2 \beta) \). The corresponding equilibrium profit is

\[ \pi_2 = 2 p_{1A} + 2 \beta V_L. \] (16)

Whereas \( \pi_2 \) exceeds \( \pi_{III} \) (i.e., \( 0 < 2 \beta < 1 \)) if and only if

\[ \rho > \frac{2V_L - V_H}{V_L}, 0 \leq \rho < 1. \] (17)

When \( 0 < 2 \beta < 1 \) we have

\[ 2 \beta = \left( \frac{(1 - \rho)V_L}{V_H - V_L} \right)^{1-\rho}, 0 < \rho < 1. \] (18)

\[ 2 p_{1A} = V_H - 2 \beta \frac{1-\rho}{\rho} (V_H - V_L); \quad 2 p_{2B} = V_L. \] (19)

Surprisingly, the above expression for \( 2 \beta \) is identical to what is described by equation (10). That is, the optimal probability of sale in this case is completely independent of the attributes of Product A (i.e., \( \delta_H \) and \( \delta_L \)).

Should the seller offer product A or product B in random sales? We find that \( \pi_1 > \pi_2 \) holds true if and only if

\[ \frac{V_H - V_L}{V_H - V_L + \delta_H - \delta_L} \geq \frac{V_L}{V_L + \delta_L - c}, 0 \leq \rho \leq 1. \] (20)

Thus we find that higher risk aversion \( (\rho) \) influences the seller to offer product A rather than product B in period 2. Quite expectedly, the above also implies that a greater difference in buyers’ valuations of product B (i.e., \( V_H - V_L \)) increases the seller’s motivation to offer product A in both periods 1 and 2, whereas a greater difference in the valuation of product A (\( V_H - V_L + \delta_H - \delta_L \)) influences the firm to offer product B in period 2.

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4 This condition is developed in Appendix I (see proof of theorem 2).
5 Further evidence lends support to this argument. It can be similarly shown that an increase in \( \rho \) influences the seller to prefer \( S_2 \) over \( S_3 \). In addition, for any \( S_3 \in \Sigma_3 \) with \( 0 < \alpha^3 < 1 \) and \( 0 < \beta^3 < 1 \) it holds that \( d \alpha^3(\rho) > 0 \) and \( d \beta^3(\rho) < 0 \).
6 See §2.2.
Strategies in the category $\Sigma_3$ involve randomized selling of both products A and B. Solutions for profit-maximizing strategies in this category ($S_3$) are developed in appendix I as part of theorem 2’s proof.

Finally, it can be verified that for each one of the five aforementioned categories there exist cases in which a unique, globally optimal strategy belongs to it.

**2.4 The effects of buyers’ risk aversion on product-line efficiency**

In this section we analyze two specific model scenarios in order to argue that an increase in buyers’ risk aversion ($\rho$) can lead *either* to an increase *or* to a decrease in the extent of quality distortion by a discriminating monopolist.

**Case I: Higher buyers’ risk aversion has a moderating effect on quality distortion**

Using the framework of section 2.3, we consider a scenario with the following parameters: $V_H = 1.1; \delta_H = 3; V_L = 1; \delta_L = 1$ and $c = 0.5$.

Let us assume for a moment that buyers are risk neutral and analyze the seller’s corresponding optimal strategies. Previous research (Riley and Zeckhauser (1983), Marom and Seidmann (2005)) establishes that whenever buyers are risk neutral ($\rho=0$) there always exists an optimal strategy ($s_{RN}^*$) that does not involve randomization. In our specific case, there is an optimal strategy $s_{RN}^* \in \Sigma_{1II}$ that entails selling product A to type-H buyers in the first period (at price $p_{1A} = 4$) and selling product B with certainty ($\beta = 1$) to type-L buyers in the second period (at price $p_{2B} = 1$). In the corresponding equilibrium the seller’s profit is $\pi = 4.5$.

Let us assume now that buyers are risk averse with a degree of constant risk aversion of $\rho = 0.85$. In this case, there is an optimal strategy $s_{RA}^* \in \Sigma_3$ that entails selling product A to type-H buyers in the first period (at price $p_{1A} = 3.964$). In the second period the seller optimally randomizes its sales as follows. With probability $\alpha=0.081$ the monopolist sells product A to type-L buyers (at price $p_{2A} = 2$). With the complementary probability $\beta = 1 - \alpha = 0.919$ the firm sells product B to type-L buyers (at price $p_{2B} = 1$). In the corresponding equilibrium, the seller’s profit is $\pi = 4.505$.

We observe that in this case a higher level of buyers’ risk aversion influences the seller to reduce quality distortion by offering the efficient product A with some likelihood in the second period to type-L buyers (whereas in $s_{RN}^*$ only the inefficient product B was offered to them). Interestingly, the increase in $\rho$ in this case also results in a Pareto improvement due to the decrease in the price $p_{1A}$ paid by type-H buyers at the equilibrium corresponding to $s_{RA}^*$(!).
Case II: Higher buyers’ risk aversion exacerbates quality distortion

We consider the following model settings: $V_H = 1.03; \delta_H = 1.45; V_L = 1; \delta_L = 1$ and $c = 0.5$.

If buyers are risk neutral ($\rho = 0$) then there is an optimal strategy $s_{RN}^* \in \Sigma_H$ that entails selling the high product A to both buyer types H and L at price $p_{1A} = 2$. The seller’s resulting profit is $\pi = 3$.

If buyers are risk averse with $\rho = 0.85$ the optimal strategy $s_{RA}^* \in \Sigma_3$ entails selling product A to type-H buyers in the first period (at price $p_{1A} = 2.287$). In the second period the monopolist sells product A to type-L buyers (at price $p_{2A} = 2$) with probability $\alpha = 0.622$, and product B (at price $p_{2B} = 1$) with a complementary probability $\beta = 1 - \alpha = 0.378$. The seller’s resulting profit is $\pi = 3.099$.

In case II, therefore, the monopolist will distort the provision of quality by introducing the inefficiently low quality product B only if buyers are risk averse.

Finally, the analysis of this section leads us to conclude that in general, a higher degree of buyers’ risk aversion has an indeterminate effect on product-line efficiency.

3. Concluding remarks

This paper extends the framework of Marom and Seidmann (2005) by considering two products rather than one. Even though the two models take different approaches and largely deal with separate issues, the main conclusions that arise from them are remarkably similar. Marom and Seidmann (2005) argue that when the monopolist’s available capacity is sufficiently high she has a unique optimal strategy that entails fixing both prices $p_1$ and $p_2$ and holding a “sale” with a probability $\alpha$ less than one. In this paper we show that it is always optimal for the seller to commit to selling but a single product (either A or B) in a random sale event (that is held with a probability less than one). Put together, the two models indicate that a monopolist cannot benefit by offering a non-degenerate joint-distribution of qualities and prices in a random sale event.

4. References


Appendix I: Proofs

**Lemma 1:** Consider a strategy \( S_0 = (0p_{1A}, 0p_{1B}, 0p_{2A}, 0p_{2B}; 0\alpha, 0\beta, 0\gamma) \) with profits \( \Pi(S_0) = \pi_0 \). If at the corresponding equilibrium all customers who buy do it in the same period, the seller can trivially attain or exceed \( \pi_0 \) with some strategy \( S_1 \) where \( 1\alpha = 1\beta = 1\gamma = 0 \). Hence it suffices to consider the case where one customer type (H or L) buys in period 1 and the other type buys in period 2. If at equilibrium type-L customers buy in period 1 and type-H customers buy in period 2 then there is a strategy \( S_2 \) with \( 2p_{2A} = V_H + \delta_H, 2p_{2B} = V_H \) and \( 2\alpha = 1 \) such that \( \pi_2 \geq \pi_0 \). Thus there exists a strategy \( S_3 \) with profits \( \pi_3 = \pi_2 \geq \pi_0 \) such that at the corresponding equilibrium H customers buy in period 1 and L customers buy in period 2. This proves the lemma’s assertion. \( \Box \)

**Lemma 2:** Let \( S_0 = (0p_{1A}, 0p_{1B}, 0p_{2A}, 0p_{2B}; 0\alpha, 0\beta, 0\gamma) \) be a strategy with \( 0\gamma > 0 \) and profits \( \Pi(S_0) = \pi_0 \).

Following lemma 1 we let,

\[
0p_{1A} = V_H + \delta_H - (0\alpha + 0\gamma)(V_H + \delta_H - 0p_{2A})^{1-\rho}
\]

\[
+ 0\beta(V_H - 0p_{2B})^{1-\rho} \cdot \frac{1}{1-\rho},
\]

\[
0p_{2A} = V_L + \delta_L \quad \text{and} \quad 0p_{2B} = V_L.
\]  

Consider now a modified strategy \( \hat{S}_0 = (0p_{1A}, 0p_{1B}, 0p_{2A}, 0p_{2B}; \hat{\alpha}, 0\beta, \hat{\gamma}) \) with \( \hat{\alpha} = 0\alpha + 0\gamma \) and \( \hat{\gamma} = 0 \). It can be easily verified that \( \hat{S}_0 \) dominates \( S_0 \). We conclude that the seller can indeed restrict herself to \( \gamma = 0 \) with no loss of optimality. \( \Box \)

**Theorem 1:** Let \( S_0 = (0p_{1A}, 0p_{1B}, 0p_{2A}, 0p_{2B}; 0\alpha, 0\beta, 0\gamma) \) be a strategy such that at the corresponding equilibrium only type-H customers are buying (in period 1). We denote \( \Pi(S_0) = \pi_0 \).

For any \( 0 \leq \beta < 1 - 1\alpha \) let the strategy \( S_1(\beta) \) be defined as follows:

\[
\hat{S}_1(\beta) = (1p_{1A}(\beta), 1p_{1B}, 1p_{2A}, 1p_{2B}; 1\alpha, 1\beta, 1\gamma), \quad \text{with},
\]

\[
1p_{1A}(\beta) = V_H + \delta_H - \beta^{1-\rho}(V_H - V_L) = 1p_{1A}, \quad 1p_{2B} = V_L,
\]

\[
1p_{1B} = V_H + \varepsilon; \quad (\varepsilon > 0), \quad \text{and} \quad 1\alpha = 1\gamma = 0.
\]
\[ \Pi(S_1(\beta)) = \Pi_1(\beta) = V_H + \delta_H - \beta^{1-\rho}(V_H - V_L) + \beta V_L - c, \quad 0 \leq \rho < 1. \]  

(26)

The above function is continuously differentiable with respect to \( \beta \). Since \( \Pi_1(0) = \pi_0 \) and \( d\Pi_1(0) = V_L > 0 \) there exists \( \beta > 0 \) such that \( \Pi_1(\beta) > \pi_0 \). Hence any optimal strategy induces type-L customers to buy with positive probability. \( \Box \)

**Theorem 2:** We need to show that for any model setting \( \{V_H, V_L, \delta_H, \delta_L, \rho, c\} \) there is an optimal strategy with either \( \alpha = 0, \beta = 0 \) or \( 1 - \alpha - \beta = 0 \).

Let \( S = (p_{1A}, p_{1B}, p_{2A}, p_{2B}; \alpha, \beta, \gamma) \). Following lemmas 1 and 2 the following optimization is equivalent to the seller’s profit-maximization:

\[
\max_S p_{1A} + \alpha \cdot p_{2A} + \beta \cdot p_{2B} - (1 + \alpha)c. 
\]  

(27)

s.t.: 

\[
(V_H + \delta_H - p_{1A})^{1-\rho} \geq \alpha(V_H + \delta_H - p_{2A})^{1-\rho} + \beta(V_H - p_{2B})^{1-\rho}. 
\]  

(28)

\[
V_L + \delta_L \geq p_{2A}. 
\]  

(29)

\[
V_L \geq p_{2B}. 
\]  

(30)

\[
\alpha + \beta \leq 1. 
\]  

(31)

\[
\alpha, \beta \geq 0. 
\]  

(32)

Condition (28) is an *incentive-compatibility* constraint for H-type buyers. Without loss of optimality, we can consider only strategies \( S \) such that at the corresponding equilibrium condition (28) is binding. That is,

\[
p_{1A} = V_H + \delta_H - (\alpha(V_H + \delta_H - p_{2A})^{1-\rho} + \beta(V_H - p_{2B})^{1-\rho})^{1-\rho}. 
\]  

(33)

Conditions (29) and (30) are *individual-rationality* constraints for L-type buyers. Because at least one of these two constraints must be binding in equilibrium we can restrict our attention to strategies \( S \) with,

\[
p_{2A} = V_L + \delta_L \quad \text{and} \quad p_{2B} = V_L. 
\]  

(34)

Let \( \lambda \) be the Lagrange multiplier that corresponds to the constraint \( \alpha + \beta \leq 1 \) and let \( \lambda_2 \) and \( \lambda_3 \) \((\lambda_2, \lambda_3 \leq 0)\) be the multipliers that correspond to \((\alpha \geq 0)\) and \((\beta \geq 0)\), respectively.
We write,

\[ L(\alpha, \beta; \lambda_1, \lambda_2, \lambda_3) = V_H + \delta_H - (\alpha(V_H - V_L + \delta_H - \delta_L)^{1-\rho} + \beta(V_H - V_L)^{1-\rho \frac{1}{1-\rho}}) + \alpha(V_L + \delta_L + \beta \cdot V_L - (1 + \alpha)c - \lambda(1 - \alpha - \beta) - \lambda_2 \cdot \alpha - \lambda_3 \cdot \beta. \]  

(35)

First-order conditions for optima include:

\[ \frac{\partial L(\cdot)}{\partial \alpha} = V_L + \delta_L - c \]

(36)

\[ \frac{\partial L(\cdot)}{\partial \beta} = V_L - \frac{(V_H - V_L)^{1-\rho} \cdot (\alpha(V_H - V_L + \delta_H - \delta_L)^{1-\rho} + \beta(V_H - V_L)^{1-\rho \frac{1}{1-\rho}})}{1 - \rho} + \lambda - \lambda_2. \]

(37)

At any optimal point where \( \alpha > 0 \) we have,

\[ \alpha = (V_H - V_L + \delta_H - \delta_L)^{\rho-1} \cdot \left( \left( \frac{(1 - \rho)(V_L + \delta_L - c + \lambda)^{1-\rho}}{(V_H - V_L + \delta_H - \delta_L)^{1-\rho}} \right)^{-\frac{1}{\rho}} - \beta \right) \cdot (V_H - V_L)^{1-\rho}. \]

(38)

And at any optimum with \( \beta > 0 \),
\[ \beta = (V_H - V_L)^{\rho - 1} \cdot \left( \frac{1 - \rho}{(V_H - V_L)^{1 - \rho}} \right) - \alpha \]

\[ \cdot (V_H - V_L + \delta_H - \delta_L)^{1 - \rho}. \]

Whenever \( \lambda < 0 \) then also \( \alpha + \beta = 1 \) and the theorem is trivially satisfied. Let us consider then the case where \( \lambda = 0 \). We define,

\[ M = \left( \frac{V_H - V_L}{V_H - V_L + \delta_H - \delta_L} \right)^{1 - \rho} \geq 0 \quad \text{and} \quad N = \frac{V_L}{V_L + \delta_L - c} \geq 0. \]

Based on different values of \( M \) and \( N \) we deal with three separate cases.

**Case I: \( M > N \)**

If at optimum \( \alpha = 0 \) then the theorem is trivially satisfied. If \( \alpha = 1 \) then \( \beta = 0 \) and the theorem is again trivially satisfied. When \( 0 < \alpha < 1 \) we get from equations (37) and (38),

\[ \frac{\partial L(\cdot)}{\partial \beta} = (V_L + \delta_L - c)(N - M) - \lambda_3 = 0 \quad , 0 < \alpha < 1, \quad \beta \leq 1 - \alpha. \]

Since \( M > N \) condition (41) is satisfied only if \( \lambda_3 < 0 \) (i.e. only if \( \beta = 0 \)), in accord with the theorem.

**Case II: \( M < N \)**

If at the optimum \( \beta = 0 \) or \( \beta = 1 \) then the theorem is trivially satisfied. Elsewhere \( 0 < \beta < 1 \), we get from equations (36) and (39),

\[ \frac{\partial L(\cdot)}{\partial \alpha} = V_L \left( \frac{1}{N} - \frac{1}{M} \right) - \lambda_2 = 0 \quad , \alpha \leq 1 - \beta, \quad 0 < \beta < 1. \]

Since \( M < N \) condition (42) is satisfied only if \( \lambda_2 < 0 \) (i.e. only if \( \alpha = 0 \)). Therefore, there is an optimal strategy \( S \) with \( \alpha = 0 \), in accord with the theorem.

**Case III: \( M = N \)**

If at the optimum \( \alpha = 0 \) or \( \alpha = 1 \) then the theorem is trivially satisfied. For any \( \alpha \) and \( \beta \) such that \( 0 < \alpha + \beta < 1 \) we have,

\[ \Pi(S) = V_H + \delta_H - (V_H - V_L + \delta_H - \delta_L)(\alpha + \beta \cdot M) + (V_L + \delta_L - c)(\alpha + \beta \cdot N) - c \]
Suppose that $S$ is an optimal policy with $\alpha > 0$ and $\beta > 0$. Let $K = \alpha + \beta \cdot N = \alpha + \beta \cdot M < 1$. The seller’s equilibrium profit is the same for all strategies with probability $\alpha$ and $\beta$ resulting in the same value of $K$. Hence there is another optimal strategy $S_0 = (0p_{1A}, 0p_{1B}, 0p_{2A}, 0p_{2B}; 0\alpha, 0\beta, 0\gamma)$ with $0\alpha = K$ and $0\beta = 0$ such that $\Pi(S_0) = \Pi(S)$, in accord with the theorem.

This completes our proof. $\square$